TROPICAL SCHEME THEORY

3. Matroids

Matroids take "It's useful to have multiple perspectives on this thing" to a ridiculous extent.

Let E be a finite set of size n. Sometimes we will identify it with $[n] = \{1, \dots, n\}$, sometimes not.

Definition 3.1. *A matroid on E is a collection $\mathcal{I} \subset 2^E$ of independent sets satisfying*
(I1) $\emptyset \in \mathcal{I}$,
(I2) If $S \in \mathcal{I}$ and $S' \subset S$ then $S' \in \mathcal{I}$, and
(I3) If $S, T \in \mathcal{I}$ and $|S| > |T|$ then there exists $x \in S$ such that $T \cup \{x\} \in \mathcal{I}$.
If $S \notin \mathcal{I}$ we say that S is a dependent set.

(I3) is called the exchange/replacement axiom.

Example 3.2.
(1) A finite collection of vectors $\{v_e \mid e \in E\}$ in a vector space determines a matroid, where $S \in \mathcal{I}$ if and only if $\{v_e \mid e \in S\}$ is linearly independent.
(2) The matroid of a subspace V of a coordinate vector space K^E is defined by $S \in \mathcal{I}$ if and only if the coordinate functions $\{x_e \mid e \in S\}$ are linearly independent when restricted to V. Note that this an instance of (1), as $\{x_e|_V \mid e \in E\}$ is a vector arrangement in V^*.
(3) A finite collection of linear hyperplanes $\{H_e | e \in E\} \subset V$ in a finite-dimensional vector space determines a matroid, where $S \in \mathcal{I}$ if and only if codim $\bigcap_{e \in S} H_e = |S|$. (We allow the degenerate hyperplane $H_e = V$.) Again, this is an instance of (1): If H_e is the vanishing locus of a functional $x_e \in V^*$, then the matroid of the arrangement of hyperplanes $H_e \subset V$ is identical to the matroid of the arrangement of vectors $x_e \in V^*$. (We note that the choice of normal vector x_e is only determined up to multiplication by $K^\times$; however, this ambiguity does not affect the underlying matroid.)
(4) Given a finite graph G, the graphic matroid $M(G)$ is the matroid on $E(G)$ where a set of edges is independent if and only if the subgraph spanned by them is a forest.
(5) A finite subset $\{\alpha_e \mid e \in E\}$ of a field extension L/K defines an algebraic matroid. Independence here is algebraic independence over K.

Date: September 12, 2017, *Speaker*: Max Kutler, *Scribe*: Netanel Friedenberg.

(6) If $P \subset K[x_1, \dots, x_n]$ is a prime ideal, take the finite set $\{x_1, \dots, x_n\}$ in the fraction field of $K[x_1, \dots, x_n]/P$. This gives the (algebraic) matroid of P. Geometrically, $S \subseteq [n]$ is independent in the matroid of P if and only if the projection of $V(P) \subseteq \mathbb{A}^n$ onto the coordinate subspace $\mathbb{A}^S$ is dominant.

Every algebraic matroid is obtained in this way: given $\alpha_1, \dots, \alpha_n \in L$, the kernel of $K[x_1, \dots, x_n] \to L$, $x_i \mapsto \alpha_i$ is a prime ideal.

Definition 3.3. *$e \in E$ is a loop of M is $\{e\}$ is dependent. (Language coming from matroid of a graph.)*
$e, f \in E$ are parallel in M if $\{e, f\}$ is dependent. (Think about parallel edges between two vertices in a graph.)

Definition 3.4. *A matroid is simple (think: simple graph) or a combinatorial geometry if it has no loops and no parallel points.*

Example 3.5. (simple matroids)

(7) A finite collection of distinct points in a projective space, with $S \in \mathcal{I}$ if and only if the projective span of S has dimension $|S| - 1$.

(8) A finite collection of distinct hyperplanes in a projective space $\mathbb{P}^d$ with $S \in \mathcal{I}$ if and only if codim $\bigcap_{e \in S} H_e = |S|$ or $\bigcap_{e \in S} H_e = \emptyset$ and $|S| = d + 1$.

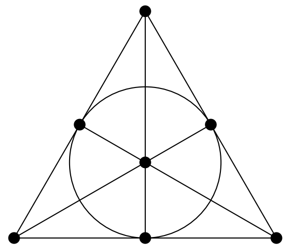
Definition 3.6. *Two matroids $(E_1, \mathcal{I}_1), (E_2 \mathcal{I}_2)$ are isomorphic if there is a bijection $f : E_1 \to E_2$ such that $S \in \mathcal{I}_1$ if and only if $f(S) \in \mathcal{I}_2$.*

Definition 3.7. *Given a field K, a matroid is realizable over K if it is isomorphic to the matroid of a vector space $V \subseteq K^E$. A matroid is realizable if it is realizable over some field. (The word representable gets used interchangeably with "realizable.") A matroid is regular if it is realizable over every field.*

Example 3.8. Every graphic matroid is regular. Why? Direct the edges, and assign each edge to the corresponding column of the adjacency matrix. Check that a collection of edges contains a cycle if and only if the corresponding columns are linearly dependent.

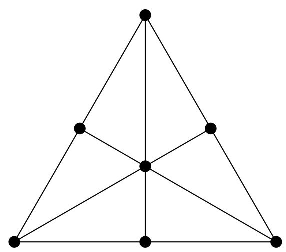
Example 3.9. The uniform matroid $U_{r,n}$ of rank r on n elements is the matroid on $[n]$ for which $S \subseteq [n]$ is independent if and only if $|S| \leq r$. The uniform matroid $U_{r,n}$ is realizable by a generic r-dimensional subspace of K^n (more precisely, $V \subset K^n$ has underlying matroid $U_{r,n}$ if and only if all Plücker coordinates of V are nonzero). However, $U_{r,n}$ need not be regular. For instance, $U_{2,4}$ is not realizable over $\mathbb{F}_2$, and $U_{2,5}$ is not realizable over $\mathbb{F}_3$.

Example 3.10. The 7 points of $\mathbb{P}^2_{\mathbb{F}_2}$ gives a matroid F_7, the Fano matroid.



The nonempty independent sets are all 1- and 2-element sets, along with the 28 sets of 3 non-collinear points. Note that taking the matroid of the lines gives an isomorphic matroid. The matroid F_7 is realizable over a field K if and only if $\mathrm{char} K = 2$.

Example 3.11. The anti-Fano matroid (sometimes denoted F_7^-) is obtained by taking the points with the same coordinates but over a field of characteristic $\neq 2$. Equivalently, take the same picture but remove the circle.



The matroid F_7^- is realizable over K if and only if $\mathrm{char} K \neq 2$.

So a good way to make a non-realizable matroid is to smash these two together.

Definition 3.12. *Given matroids $M = (E, \mathcal{I})$, $N = (F, \mathcal{J})$, the direct sum $M \oplus N$ is the matroid on $E \sqcup F$ where the independent sets are $\{I \sqcup J \mid I \in \mathcal{I}, J \in \mathcal{J}\}$.*

Example 3.13. The direct sum $F_7 \oplus F_7^-$ is not realizable over any field.

Theorem 3.14 ([Nel16])**.** *As $n \to \infty$,*

$$\frac{\#\{\textit{representable matroids on } [n]\}}{\#\{\textit{matroids on } [n]\}} \to 0.$$

Idea of proof: Knuth proved that the number of matroids on $[n]$ grows doubly exponential with n. A realization of a matroid must satisfy non-vanishing of certain determinants. By carefully examining these non-vanishing conditions, Nelson bounds the number of realizable matroids on $[n]$ (for $n \geq 12$) by $2^{n^3/4}$. The result follows.

Cryptomorphism (i.e. matroids have a million different definitions.)

Definition 3.15 (Definition 2)**.** *A matroid E is a collection of bases $\mathcal{B} \subset 2^E$ satisfying*
(B1) $\mathcal{B} \neq \emptyset$, and
(B2) If $B_1, B_2 \in \mathcal{B}$ are distinct then for any $e \in B_1$ there exists $f \in B_2 \backslash B_1$ such that $(B_1 \backslash \{e\}) \cup \{f\} \in \mathcal{B}$.

Exercise 3.16. Prove that if $\mathcal{I} \subset 2^E$ satisfies axioms (I1)–(I3), then the set $\mathcal{B}$ of inclusion-maximal sets in $\mathcal{I}$ satisfies axioms (B1)–(B2). Conversely, if $\mathcal{B} \subset 2^E$ satisfies (B1)–(B2), define $\mathcal{I} \subset 2^E$ to be the set of all subsets of elements of $\mathcal{B}$ and prove that $\mathcal{I}$ satisfies (I1)–(I3). Thus, the definition of matroids in terms of bases is equivalent to the definition of matroids in terms of independent sets.

We could axiomatize dependent sets, and formulate a definition of matroids in terms of these axioms. We won't write this down.

Just as we can axiomatize matroids in terms of maximal independent sets (bases), we can also axiomatize matroids using minimal dependent sets (circuits).

Definition 3.17 (Definition 3)**.** *A matroid on E is a collection of circuits $\mathcal{C} \subset 2^E$ satisfying*
(C1) $\emptyset \notin \mathcal{C}$,
(C2) if $C \in \mathcal{C}$, *no proper subset of* $C \in \mathcal{C}$, *and*
(C3) If $C_1, C_2 \in \mathcal{C}$ *are distinct then for any* $j \in C_1 \cap C_2$ *the set* $C_1 \cup C_2 \backslash \{j\}$ *contains a circuit.*

The intuition for (C3) is as follows. For a subspace $V \subseteq K^n$, a minimal dependent set corresponds to a collection of coordinate functions $\{x_i \mid i \in C\}$ which satisfy a unique (up to scaling by $K^\times$) linear relation $\sum_{i\in C} a_i x_i$ on V, with all $a_i \neq 0$. Given two such linear forms of minimal support, each involving x_j, we can appropriately scale and subtract to obtain a linear form not involving x_j, but this won't necessarily have minimal support.

Exercise 3.18 (Somewhat more difficult exercise)**.** Show that this definition is equivalent to the definition of matroids in terms of independent sets. As with the previous exercise, this requires not only defining $\mathcal{I}$ in terms of $\mathcal{C}$ and vice versa, but showing that axioms (I1)–(I3) are equivalent to axioms (C1)–(C3).

Definition 3.19 (Definition 4)**.** *A matroid on E is a rank function* $\mathrm{rk} : 2^E \to \mathbb{Z}_{\geq 0}$ *such that*
(R1) $\mathrm{rk}(S) \leq |S|$,
(R2) (submodularity) $\mathrm{rk}(S \cup T) + \mathrm{rk}(S \cap T) \leq \mathrm{rk}(S) + \mathrm{rk}(T)$, *and*
(R3) For all $S \subset E$ *and* $x \in E$ $\mathrm{rk}(S) \leq \mathrm{rk}(S \cup \{x\}) \leq \mathrm{rk}(S) + 1$.

The rank of a set is the size of its largest independent subset. The rank of a matroid M on E is $\mathrm{rk}(E)$, which is equal

The following is the least intuitive definition of matroids we'll see. A flat is a maximal set of a given rank.

Definition 3.20 (Definition 5)**.** *A matroid on E is a collection $\mathcal{L}$ is a collection* $\mathcal{L} \subset 2^E$ *of flats such that*
(F1) $E \in \mathcal{L}$,
(F2) if $F_1, F_2 \in \mathcal{L}$ *then* $F_1 \cap F_2 \in \mathcal{L}$, *and*
(F3) If $F \in \mathcal{L}$ *the flats that cover* F *partition* $E \backslash F$.
Covering is in the poset sense: F' *covers* F *if* $F \subsetneq F'$ *and* $F \subset S \subset F'$ *with* $S \in \mathcal{L}$ *implies* $S = F$ *or* $S = F'$.

We use the letter $\mathcal{L}$ because the collection of flats, partially ordered by inclusion, forms a geometric lattice, a lattice which is finite, submodular with respect to the rank function, and in which every element is a join of atoms (an atom is a rank 1 flat).

The smallest flat containing a set S is called the span of S, or the closure of S. There is a cryptomorphic definition of matroids in terms of the closure operator, which takes a set to its span, but we will not consider that here.

Exercise 3.21 (Tricky "highly non-intuitive" exercise)**.** Translate between flats and circuits.

Exercise 3.22 (Not as tricky, but somewhat unnatural exercise)**.** Identify the flats of a graphic matroid.

Example 3.23. We can visualize flats in the realizable case as follows.

If M is the matroid of an arrangement of hyperplanes in a finite-dimensional vector space V, then the rank of a set of hyperplanes is the codimension of their intersection. Thus, a flat is the set of all hyperplanes containing a given subspace of V.

Equivalently, if M is the matroid of a vector arrangement in V^*, then the rank of a set of vectors is the dimension of their span. A flat is therefore the set of all vectors contained in a given subspace of V^*.

Duality

We now discuss a notion of duality for matroids, which generalizes the notion of duality for planar graphs. (This generalization of duality for matroids, such as non-planar graphical matroids, that did not otherwise have duals, was one of the original motivations for the development of matroid theory.) Given a matroid M, there is a naturally constructed dual matroid M^* on the same underlying set. This construction has the property that $(M^*)^* = M$. It is worth emphasizing that the duality of matroids is not a generalization of duality of vector spaces; rather, it is a generalization of *Gale duality* of vector arrangements.

Example 3.24. Let $\{v_1, \dots, v_n\}$ be a collection of vectors in a finite-dimensional vector space V over a field K. Replacing V with $\operatorname{span}\{v_1, \dots, v_n\}$, we assume that $\{v_1, \dots, v_n\}$ spans V.

Let $W \subset K^n$ be the vector space of solutions $(\mathbf{a}) = (a_1, \dots, a_n)$ to the linear equation $\sum_{i=1}^{n} a_i v_i = 0$. This is not only a vector space, but a vector space with n functionals $w_i : W \to k, (\mathbf{a}) \mapsto a_i$. That is, we have an arrangement $\{w_1, \dots, w_n\} \subset W^*$. This is called the Gale dual arrangement of $\{v_1, \dots, v_n\} \subset V$.

Another way to think about this is that W is the kernel of the surjection $K^n \to V$, $e_i \mapsto v_i$. We have the short exact sequences

$$0 \to W \to K^n \to V \to 0,$$

which dualizes to give

$$0 \longleftarrow W^* \longleftarrow K^n \longleftarrow V^* \longleftarrow 0.$$

Note that W is the orthogonal complement of V^*, with respect to the standard inner product on K^n.

Theorem 3.25. *A subset $\{v_i \,|\, i \in S\}$ is linearly independent in V if and only if $\{w_i | i \notin S\}$ spans W^*. In particular, $\{v_i | i \in S\}$ is a basis of V if and only if $\{w_i | i \notin S\}$ is a basis of W^*.*

Exercise 3.26 (Not hard exercise)**.** Prove this theorem.

These properties of Gale duality for vector arrangements inspire the following definition of matroid duality.

Definition 3.27. *If M is a matroid on E, the dual matroid M^* is the matroid on E whose bases are complements of bases of M.*

Exercise 3.28 (Easy but nontrivial exercise)**.** Prove that M^* is a matroid. That is, show that the basis exchange axiom works. If B_1, B_2 are bases of M then for any $x \in B_1$ there exists $y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup \{y\}$ is a basis. The key point of the exercise is to show that the basis exchange axiom holds for M^*.

Example 3.29. If M is a realizable matroid, then M^* is the matroid of the Gale dual of any realization of M. This follows directly from Theorem 3.25.

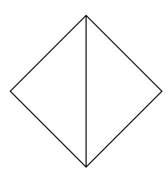
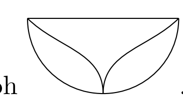
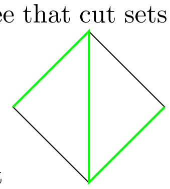
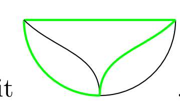
Example 3.30. The dual to a graphic matroid is called a cographic matroid. By a theorem of Whitney, a graph G is planar if and only if the graphic matroid $M = M(G)$ is also cographic, in which case M^* is the matroid of the dual graph of G.

Duality essentially doubles the number of cryptomorphic definitions of matroids. These definitions are not novel—after all, the independent sets, circuits, flats, etc. of M^* must satisfy the same axioms as those of M—but we get useful definitions when we can relate data of the dual matroid directly to data of the original matroid.

Exercise 3.31. A subset $H \subset E$ is a hyperplane of M if H is a flat of rank $\mathrm{rk}(M)-1$. Prove that H is a hyperplane of M if and only if H^c is a cocircuit of M, i.e. a circuit of M^*.

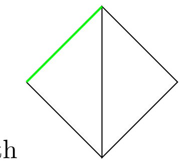
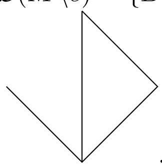
Use the circuit axioms (C1)–(C3) to provide a cryptomorphic definition of a matroid M in terms of its set of hyperplanes $\mathcal{H}$.

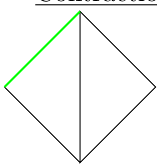
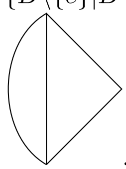
Example 3.32. In a graphic matroid, a cocircuit is a cut-set - the set of edges connecting a vertex in V_1 to a vertex in V_2 for some partition $V(G) = V_1 \sqcup V_2$ such that the induced subgraphs on V_1 and V_2 are connected.

Example 3.33. The planar graph $G =$  has dual graph . You can see that cut sets of G correspond to circuits in its planar dual. For example, the cut-set  corresponds to the circuit .

Deletion and Contraction

There are 2 ways to make a matroid on $E\backslash\{e\}$ from M.

Deletion: $\mathcal{B}(M\backslash e) = \{B \in \mathcal{B}(M) | e \notin B\}$. Think about starting with  and going to .

Contraction: $\mathcal{B}(M/e) = \{B\backslash\{e\} | B \in \mathcal{B}(M), e \in B\}$. Think about starting with  and going to .

In terms of vectors, think about removing a vector and projecting, respectively.

A minor of M is a matroid obtained by a sequence of deletions and contractions. It is often the case that specific minors form an obstruction to M possessing some structure. For instance,

Theorem 3.34 (Tutte)**.** *A matroid M is regular if and only if M does not contain $U_{2,4}$, F_7, or $(F_7)^*$ as a minor.*

Many arguments/constructions in matroid theory go by induction/recursion on deletion on contraction steps to reduce to a minor. Examples are the chromatic polynomial and the Tutte polynomial.

The Bergman fan of a matroid

Let M be a loopless matroid on $[n]$. The Bergman fan of M, denoted $\widetilde{B}(M)$ is the set

$$\left\{(w_1, \ldots, w_n) \in \mathbb{R}^n \,\middle|\, \min_{i \in C}\{w_i\} \text{ is attained at least twice for each circuit } C\right\}.$$

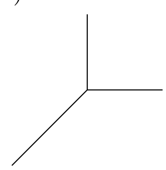
Where does this come from? If M is the matroid of a linear subspace $V \subseteq K^n$, then the condition that M have no loops is equivalent to V not being contained in a coordinate subspace, so that $V \cap (K^\times)^n$ is nonempty. Each circuit C of M comes from a linear functional of minimal support C, $l_C = \sum_{i \in C} a_i x_i$, unique up to scaling by $K^\times$. If we equip K with the trivial valuation, then the tropicalization of the equation l_C is the condition that $\min_{i \in C}\{w_i\}$ is attained at least twice. It is a theorem that the set $\{l_C \,|\, C \text{ is a circuit}\}$ is a tropical basis of the ideal of V. It follows that, for realizable M, the Bergman fan $\widetilde{B}(M)$ is equal to $\mathrm{Trop}(V)$ for any realization V. Of course, it is possible to define $\widetilde{B}(M)$ for a non-realizable matroid M, and it is an interesting problem (which we will consider in more detail below) to ask if $\widetilde{B}(M) = \mathrm{Trop}(X)$ has a solution $X \subset (K^\times)^n$.

Note: It follows from the definition that $\widetilde{B}(M) + \mathbb{R}(1, \ldots, 1) = \widetilde{B}(M)$. We will write $B(M) = \widetilde{B}(M)/\mathbb{R}(1, \ldots, 1)$. Some authors define the Bergman fan to be $B(M)$, but the language is not standard. We will try to distinguish by calling $B(M)$ the reduced Bergman fan.

Example 3.35. If $M = U_{2,3}$ is the uniform matroid of rank 2 on 3 elements, then $\{1, 2, 3\}$ is the only circuit. It follows that the Bergman fan is

$$\widetilde{B}(M) = \{(w_1, w_2, w_3) \mid \min\{w_1, w_2, w_3\} \text{ is attained at least twice}\},$$

and the reduced Bergman fan $B(M)$ is



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It is possible to define $\widetilde{B}(M)$ in terms of the other data of a matroid. We'll specifically look at bases and flats.

For $w \in \mathbb{R}^n$, the w-weight of a basis B is $\sum_{i \in B} w_i$. Note that for generic w, there will be very few bases of maximal w-weight.

Definition 3.36. *Given M and $w \in \mathbb{R}^n$, let M_w be the matroid on $[n]$ with bases the w-maximal bases of M.*

(Note: Maximal is correct here; if we use min with circuits, then we must use max with bases.)

Theorem 3.37. *A weight w is in $\widetilde{B}(M)$ if and only if M_w has no loops.*

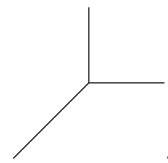
Proof. If $w \notin \widetilde{B}(M)$, then there is some circuit C and $i \in C$ with

$$w_i < \min\{w_j \mid j \in C \backslash \{i\}\}.$$

Now, suppose i is in some basis B of M. The span of $C\backslash\{i\}$ contains i, and so $(B \cup C)\backslash\{i\}$ spans the entire set $[n]$. That is, there exists $j \in C\backslash\{i\}$ such that $(B\backslash\{i\}) \cup \{j\}$ is a basis. This basis has strictly greater w-weight than B, and so i is a loop in M_w.

Conversely, if i is a loop in M_w then for any w-maximizing basis B, $B \cup \{i\}$ contains a circuit C of M. Necessarily, $i \in C$, since $C \not\subset B$. For $j \in B$ we have that $(B\backslash\{j\}) \cup \{i\}$ is a basis if and only if $j \in C\backslash\{i\}$. But since i is not contained in any w-maximal basis, $(B\backslash\{j\}) \cup \{i\}$ is not w-maximal. Therefore, $w_i < w_j$. Since this holds for every $j \in C\backslash\{i\}$, we have $w_i < \min\{w_j \mid j \in C\backslash\{i\}\}$ and $w \notin \widetilde{B}(M)$. □

Example 3.38. Recall that for $M = U_{2,3}$, the reduced Bergman fan $B(M)$ is the following subset of $\mathbb{R}^3/\mathbb{R} \cdot (1,1,1)$.



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We consider two of the relevant regions. The bottom left ray of $B(M)$ is given by weight vectors (w_1, w_2, w_3) such that $w_1 = w_2 \leq w_3$. In this case the w-maximal bases are $\{1,3\}$ and $\{2,3\}$, so M_w has no loops. Thus every point on this ray is in $B(M)$. Similar analysis applies to the other rays.

The open first quadrant in the picture is defined by $w_3 < w_1, w_2$. For any such weight vector, the only w-maximal basis of M is $\{1,2\}$, so 3 is a loop of M_w. Thus no such w is in $B(M)$. The two other regions in the complement of $B(M)$ come from weight vectors for which 1 or 2 is a loop in M_w.

For $w \in \mathbb{R}^n$ let
$$\mathcal{F}(w) = \{\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq [n]\}$$
be the unique flag of subsets with
•w constant on each $F_{i+1} \backslash F_i$ and
•$w|_{F_i \backslash F_{i-1}} > w|_{F_{i+1} \backslash F_i}$.

Example 3.39. For $w = (1,7,1,1,2,7)$, $F(w) = \{\emptyset \subsetneq \{2,6\} \subsetneq \{2,5,6\} \subsetneq [n]\}$.

Lemma 3.40 ([AK06, Proposition 1]). *The matroid M_w depends only on $\mathcal{F}(w)$.*

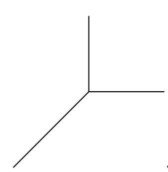
The idea of the proof is to use the greedy algorithm. A subset of $[n]$ is a w-maximal basis if and only if it contains precisely $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1})$ elements of $F_i \backslash F_{i-1}$ for each i.

Theorem 3.41 ([AK06, Theorem 1]). *A weight vector w is in $\widetilde{B}(M)$ if and only if $\mathcal{F}(w)$ is a flag of flats.*

Proof. If F_i is not a flat, then there is some $e \notin F_i$ such that e is in the span of F_i. In the process of constructing a w-maximal basis, the greedy algorithm will select $\mathrm{rk}(F_i)$ elements of F_i. Since e is in the span of these elements, it cannot be in the resulting basis. Thus, e is a loop of M_w, so $w \notin \widetilde{B}(M)$.

Conversely, if each F_i is a flat, then for any e there is an index i such that $e \in F_i \backslash F_{i-1}$. In constructing a w-maximal basis, the greedy algorithm is free to pick e among the $\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1})$ elements that it selects from $F_i \backslash F_{i-1}$. Therefore, M_w contains no loops. □

Example 3.42. Once again, recall that for $M = U_{2,3}$ we have that $B(M)$ is



.

We consider the same two regions as before. The bottom left ray is given by $w_1 = w_2 < w_3$. In this case we have $\mathcal{F}(w) = \{\emptyset \subsetneq \{3\} \subsetneq \{1,2,3\}\}$ which is a flag of flats of $U_{2,3}$. So every point on they ray is in $B(M)$.

The open first quadrant in the picture is given by $w_3 < w_1, w_2$. In this case $\mathcal{F}(w)$ is one of $\{\emptyset \subsetneq \{1,2\} \subsetneq \{1,2,3\}\}$, $\{\emptyset \subsetneq \{1\} \subsetneq \{1,2\} \subsetneq \{1,2,3\}\}$, or $\{\emptyset \subsetneq \{2\} \subsetneq \{1,2\} \subsetneq \{1,2,3\}\}$. Since $\{1,2\}$ is not a flat of $U_{2,3}$ this shows that no such w is in $B(M)$.

This gives us 2 fan structures on M:
Fine: w and v are in the same cone if and only if $\mathcal{F}(w) = \mathcal{F}(v)$. The cone corresponding to a flag $\mathcal{F}$ is

$$C_{\mathcal{F}} = \mathbb{R}_{\geq 0} \langle u_{F_1}, u_{F_2} \dots, u_{F_k} \rangle + \mathbb{R}(1,1,\dots,1)$$

where $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq [n])$.
Coarse: w and v are in the same cone if and only if $M_w = M_v$.
This is in fact coarser - by Lemma 3.40, once you know $\mathcal{F}(w)$, you know M_w.

Matroids in tropical geometry

Example 3.43. Consider the uniform matroid $U_{3,4}$ and its reduced Bergman fan $B(U_{3,4})$. The coarse structure on $B(U_{3,4})$ is the 2-skeleton of the fan of $\mathbb{P}^3$. In the fine structure we subdivide the maximal cones: in one example, M_w has bases $123, 124$, and in the fine structure we care which of $1, 2$ has more weight, corresponding to the flag being $\emptyset \subset 2 \subset 12 \subset 1234$ or $\emptyset \subset 1 \subset 12 \subset 1234$.

Theorem 3.44 ([Huh14, Theorem 38])**.** *The reduced Bergman fan of a matroid is balanced if and only if all maximal cones have equal weight.*

Proof. Set $r = \mathrm{rk}(M)$. Let $\tau = C_{\mathcal{F}}$ be a codimension 1 cone. Write $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-2} \subsetneq [n])$ for the corresponding flag of flats. There is precisely one index l such that $\mathrm{rk}(F_l) = \mathrm{rk}(F_{l-1}) + 2$. Let $G_1, \dots, G_m$ be the flats of rank $\mathrm{rk}(F_{l-1}) + 1$ such that $F_{l-1} \subsetneq G_i \subsetneq F_l$. Let $\mathcal{F}_i$ be the maximal flag obtained from $\mathcal{F}$ by inserting G_i. So

$$\mathcal{F}_i = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{l-1} \subsetneq G_i \subsetneq F_l \subsetneq \cdots \subsetneq F_{r-2} \subsetneq [n]).$$

The balancing condition at τ says that $\displaystyle\sum_{i=1}^{m} \mathrm{wt}(C_{\mathcal{F}_i}) u_{G_i} \in \mathbb{R}\tau$. So

$$\sum_{i=1}^{m} \mathrm{wt}(C_{\mathcal{F}_i}) u_{G_i \setminus F_l} = c_1 u_{F_1} + c_2 u_{F_2 \setminus F_1} + \cdots + c_{r-2} u_{F_{r-2} \setminus F_{r-3}} + c_{r-1} u_{[n] \setminus F_{r-2}}.$$

The flat axioms applied to the restriction of the matroid to F_l tell us that $F_l \setminus F_{l-1} = \bigsqcup G_i \setminus F_{l-1}$. So the support of the vector $\displaystyle\sum_{i=1}^{m} \mathrm{wt}(C_{\mathcal{F}_i}) u_{G_i \setminus F_l}$ is $F_l \setminus F_{l-1}$. Hence $c_l = \mathrm{wt}(C_{\mathcal{F}_i})$ for all i.

It is a theorem of Björner [Bjo92] that the order complex of the lattice of flats of a matroid is shellable. As a corollary, $B(M)$ is connected in codimension one. Since the above argument applies to every codimension one cone τ, it follows that every maximal cone has the same weight. $\square$

Realizability

Given a loopless matroid M does there exist a field K and $X \subset \mathbb{G}_{m,K}^n$ with $\mathrm{Trop}(X) = \widetilde{B}(M)$? We follow the approach of [Yu17].

Reminder: An algebraic matroid comes from $\{\alpha_1, \ldots, \alpha_n\} \in L/K$ Equivalently, it is given by a prime ideal $P \subset K[x_1, \ldots, x_n]$. This is nice because we can turn it into algebraic geometry by setting $X = V(P)$. Now S is independent if and only if the coordinate projection $X \to \mathbb{A}^S$ is dominant.

Can we do the same thing with tropical varieties?

Assume P is monomial-free and let $X' = X \bigcap \mathbb{G}_m^n$.

Lemma 3.45 ([Yu17, Lemma 2]). *$S \subset E$ is independent in the algebraic matroid of P if and only if the projection of $\mathrm{Trop}(X')$ onto the coordinate subspace $\mathbb{R}^S$ has dimension $|S|$ (from which it follows that the projection must be all of $\mathbb{R}^S$).*

Proof. Move to a valued extension such that tropicalization is surjective. Then this is immediate from the algebraic geometry version. Note that passing to a valued extension changes neither the algebraic matroid nor the tropicalization. $\square$

Question 3.46 (Open question). *For a tropical variety, when do we get a matroid from coordinate projections?*
Can this lead to/inform a good notion of irreducible/prime ideals/congruences?

There is a partial answer:

Lemma 3.47 ([Yu17, Lemma 3]). *For any loop-free matroid M, $S \subset E$ is independent in M if and only if the projection of $\widetilde{B}(M)$ onto $\mathbb{R}^S$ has dimension $|S|$.*

Proof. If $S = \{s_1, \ldots, s_k\}$ is independent, make a flag

$$\emptyset \subsetneq \mathrm{span}\{s_1\} \subsetneq \mathrm{span}\{s_1, s_2\} \subsetneq \cdots \subsetneq \mathrm{span} S.$$

The corresponding cone has full dimension upon projection to $\mathbb{R}^S$.
Conversely, if S is dependent, let

$$\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E)$$

be any flag of flats. In the matroid obtained by restricting to S (equivalently, by deleting S^c) we get a flag of flats

$$\emptyset \subset F_1 \cap S \subset \cdots \subset F_k \cap S \subsetneq S.$$

Because $\mathrm{rk}(S) < |S|$, there are strictly fewer than $|S|$ flats here. So when we project to $\mathbb{R}^S$ we get the span of fewer than $|S|$ things, and so it can't have dimension $|S|$.

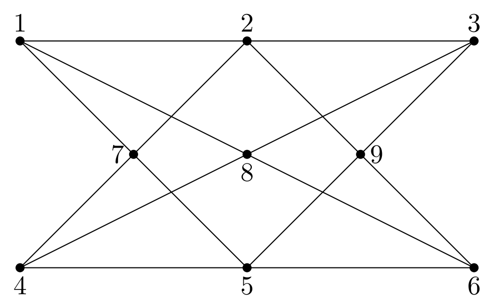
Since this is true for all $\mathcal{F}$, we see that the whole projection has dimension less than $|S|$. □

Theorem 3.48 ([Yu17, Theorem 4])**.** *If* $\widetilde{B}(M) = \mathrm{Trop}(X)$ *for some* $X \subset \mathbb{G}^n_{m,K}$ *then* M *is algebraic over* K*.*

Proof. Suppose $\widetilde{B}(M) = \mathrm{Trop}(X)$. Since $\widetilde{B}(M)$ can only be balanced when all cones have equal weight, it cannot contain proper tropical subvarieties of full dimension. Thus, $\widetilde{B}(M)$ is the tropicalization of every top-dimensional irreducible component of X. That is, if $\widetilde{B}(M)$ is realizable, then it is realizable by an irreducible variety. By Lemmas 3.45 and 3.47, M is algebraic. □

Question 3.49 (Open question)**.** *Is the converse of this theorem true?*

The smallest example of an algebraic matroid that isn't linearly realizable is the non-Pappus matroid. This is the matroid given by considering a set of points as in Pappus's theorem (see `https://en.wikipedia.org/wiki/Pappus%27s_hexagon_theorem`), except that the 3 points which Pappus's theorem says will be collinear are declared to not be collinear, so we automatically get that this is not linearly realizable over any field. An alternative description of this matroid is as the matroid on $\{1, \ldots, 9\}$ whose bases are the sets of three points not on a straight line drawn in the following picture:



This matroid is, however, algebraic over every finite field. For some realizations see [Ros14, Example 3.5] (this paper may be of independent interest). A starting point for exploring the above open question might be to test it for this matroid.

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