# TROPICAL SCHEME THEORY

## 4. Matroids over partial hyperstructures

The current lecture is based on [BB16].

**Definition 4.1.** *A* tract *is an abelian group $G$ (written multiplicatively) together with a subset $N_G \subset \mathbb{N}[G]$ satisfying:*
(T0) $0 \in N_G$,
(T1) $1 \notin N_G$,
(T2) *For each $g \in G$ there is some $(-g) \in G$ such that $g + (-g) \in N_G$*
(T3) *If $g \in G$ and $x \in N_G$ then $g \cdot x \in N_G$.*

**Remark 4.2.** The letter $N$ stands for "null", and $N_G$ may be thought of as a designated set of formal sums that are considered equivalent to zero in $\mathbb{N}[G]$. Since we cannot quotient by $N_G$, we simply carry around this extra data. Then roughly speaking, (T2) says that, modulo $N_G$, we have additive inverses in $G$, for every element of $G$. (Given (T3), the axiom (T2) can be replaced by the weaker requirement there is some $(-1) \in G$ such that $1 + (-1) \in N_G$.)

Recall that a *hyperoperation* is like a binary operation that can be multi-valued.

**Definition 4.3.** *A* hyperfield *is an abelian group $(F^\times, \cdot, 1)$ together with an associative and commutative* hyperoperation *$\boxplus$ on $F = F^\times \cup \{0\}$ satisfying*
- $0 \neq 1$,
- $0 \boxplus x = \{x\}$,
- $0 \cdot x = x \cdot 0 = 0$,
- $a \cdot (x \boxplus y) = ax \boxplus ay$,
- $(\forall x \in F)(\exists! -x \in F)$ *such that* $0 \in x \boxplus -x$.

**Example 4.4.** Given a hyperfield $F$ we can associate to it a tract by setting $G = F^\times$ and $N_G = \{\sum a_i g_i \in \mathbb{N}[G] \,|\, 0 \in \boxplus_i a_i g_i\}$. We can recover the hyperfield from the tract because we have $x \in \boxplus y_i$ if and only if $0 \in -x \boxplus (\boxplus y_i)$. Moreover, there are natural categories of hyperfields and tracts and the category of hyperfields embeds as a full subcategory of the category of tracts.

**Example 4.5.** Any field $F$ is a hyperfield.

**Example 4.6.** The Krasner hyperfield is $\mathbb{K} = \{0, 1\}$, with the usual multiplication and $\boxplus$ given by

| $\boxplus$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | $\{0, 1\}$ |

Think of the hyperfield as reflecting arithmetic in a field, but only keeping track of "zero or non-zero".

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**Example 4.7.** The tropical hyperfield is $\mathbb{T} = \mathbb{R} \cup \{+\infty\}$, with multiplication being $+$ and $\boxplus$ given by $x \boxplus y = \begin{cases} \min(x,y) & \text{if } x \neq y \\ \{\geq \min\} & \text{if } x = y \end{cases}$. Think of this as reflecting arithmetic in a valued field where $\mathbb{R}$ is the value group, but only keeping track of the valuations.

**Example 4.8.** The hyperfield of signs is $\mathbb{S} = \{0, +, -\}$ with the usual multiplication and $\boxplus$ given by

| $\boxplus$ | $0$ | $+$ | $-$ |
|---|---|---|---|
| $0$ | $0$ | $+$ | $-$ |
| $+$ | $+$ | $+$ | $\{0,+,-\}$ |
| $-$ | $-$ | $\{0,+,-\}$ | $-$ |

Think of this as reflecting arithmetic in $\mathbb{R}$, but only keeping track of the signs.

Moving from signs in $\mathbb{R}$ to arguments in $\mathbb{C}$ we also get the phase hyperfield.

**Example 4.9.** The phase hyperfield is $\mathbb{P} = S^1 \cup \{0\}$, where $S^1$ is the unit circle in $\mathbb{C}$. The addition is given by $x \boxplus y := \left\{ \frac{ax+by}{|ax+by|} \mid a, b \in \mathbb{R}_{>0} \right\}$ and multiplication is the group law on $S^1$, inherited from multiplication in $\mathbb{C}$.

We will define a notion of a <u>$F$-matroid</u> of rank $r$ on $E = \{1, 2, \ldots, m\}$ for any tract $F$. This will have the properties:

- If $F$ is a field, $F$-matroids correspond to $W \subset F^m$ an $r$-dimensional linear subspace (or equivalently, a point on a Grassmannian).
- If $F = \mathbb{K}$, a $\mathbb{K}$-matroid is the same as a matroid.
- If $F = \mathbb{T}$, we get the notion of valuated matroids. (These will correspond to tropical linear spaces.)
- If $F = \mathbb{S}$ we get the notion of oriented matroid.
- If $F = \mathbb{P}$ we get the phase matroids.

We also mention <u>partial fields</u> (due to Semple-Whittle), as another source of interesting tracts. They were originally defined for purely matroid-theoretic reasons.

**Definition 4.10.** *A <u>partial field</u> $F$ is a pair $(R, G)$ where $R$ is a commutative ring with* 1 *and $G \leq R^\times$ such that $-1 \in G$ and $G$ generates $R$ as a ring. This is called a partial field because in $G$ you can do everything except add arbitrary elements of $G$. If we write $F$ as a set we mean $G \cup \{0\}$.*

Note that while in a hyperfield you can add any pair, you just might get multiple answers, here the addition is single-valued but might not always exist. You can combine these ideas to get a "partial hyperfield".

**Example 4.11.** The tract associated to a partial field $F$ is given by

$$N_G = \left\{ \sum a_i g_i \,\middle|\, \sum a_i g_i = 0 \text{ in R} \right\}.$$

**Remark 4.12.** Note that there is a pre-existing "classical" notion of matroid representable over a partial field. If $F$ is a partial field then an $F$-matroid in the sense we will introduce shortly (using tracts) is the same as a matroid representable over $F$ in the classical sense.

**Example 4.13** (Partial fields)**.** $R = \mathbb{Z}$, $G = \langle \pm 1 \rangle = \mathbb{Z}^\times$. So $F = \{0, 1, -1\}$. This is often written $\mathbb{U}$ and called the regular partial field because it turns out that a $\mathbb{U}$-matroid is the same thing as a regular matroid. A matroid is regular if it is representable over every field.

**Remark 4.14.** All graphical matroids are regular. Duals of regular matroids are regular. But duals of graphical matroids are not necessarily graphical. In fact Whitney proved that the dual of a graphical matroid is graphical if and only if the graph is planar, and then the dual matroid is the matroid of any planar dual graph.

**Remark 4.15.** The notion of the Jacobian of a graph can be extended to regular matroids, and many of the theorems extend.

**Example 4.16.** The dyadic hyperfield $\mathbb{D}$ contists of the ring $\mathbb{Z}[\frac{1}{2}]$ with $G = \{\pm 2^k \mid k \in \mathbb{Z}\}$. This is called $\mathbb{D}$ the dyadic hyperfield. A $\mathbb{D}$-matroid is the same as a matroid which is representable over every field of characteristic $\neq 2$.

**Definition 4.17.** *A strong* <u>*$F$-matroid*</u> *of rank $r$ on $E = \{1, \ldots, m\}$ is an equivalence class of functions $\varphi : E^r \to F = G \cup \{0\}$ such that*

- *$\varphi \not\equiv 0$,*
- *$\varphi$ is alternating: swapping two elements negates the value and having two coordinates the same gives a value of 0, and*
- *(Plücker relations) $(\forall \{x_1, \ldots, x_{r+1}\}, \{y_1, \ldots, y_{r-1}\} \subset E)$,*

$$\sum_{k=1}^{r+1} (-1)^k \varphi(x_1, \ldots, \hat{x_k}, \ldots, x_{r+1}) \varphi(x_k, y_1, \ldots, y_{r-1}) \in N_G.$$

*Here the equivalence relation is $\varphi_1 \sim \varphi_2$ if and only if $\exists g \in G$ such that $\varphi_1 = g \cdot \varphi_2$. (This corresponds to looking at the Plücker relations in projective space rather than affine space.)*

Idea: If $F$ was a field, write down a $r \times m$ nonzero matrix and take the determinants of the $r \times r$ minors. This records the abstract algebraic properties that such determinants would have.

**Definition 4.18.** *A* <u>*weak $F$-matroid*</u> *of rank $r$ on $E = \{1, \ldots, m\}$ is an equivalence class of functions $\varphi : E^r \to F = G \cup \{0\}$ such that*

- *$\varphi \not\equiv 0$,*
- *$\varphi$ is alternating: swapping two elements negates the value (and having two coordinates the same gives a value of 0),*
- supp $\varphi$ *is the set of bases of a matroid, and*
- *The 3-term Plücker relations are satisfied.*

Note that the full Plücker relations imply that supp $\varphi$ is the set of bases of a matroid. Looking over the Krasner hyperfield, you can see that the Plücker relations essentially say that we have the basis exchange axiom.

There is a special class of tracts given by doubly distributive (partial) hyperfields: (we will only do the doubly distributive hyperfield case)

$$(\boxplus x_i) \cdot (\boxplus y_j) = \boxplus x_i y_j.$$

**Example 4.19.** Many of the examples we saw earlier are double distributive. In particular, $\mathbb{K}$, $\mathbb{T}$, $\mathbb{S}$, and partial fields are doubly distributive.

**Example 4.20** (non-examples)**.** The phase hyperfield $\mathbb{P}$ is not doubly distributive. The "triangle hyperfield" in which $a \boxplus b = \{$all $c$ such there is a Euclidean triangle with side-lengths $a, b, c\}$ is also not doubly distributive.

**Theorem 4.21.** *If $F$ is a doubly distributive partial hyperfield then weak $F$-matroids are strong.*

There are 2 other cryptomorphic characterizations of both weak and strong matroids over tracts: one given by by circuit elimination, and the other in terms of "dual pairs".

One of the original reasons for considering matroids in the first place is to have a suitable notion of duals for objects that don't classically have them, such as non-planar graphs.

**Definition 4.22.** *Let $M$ be an $F$-matroid. Define the dual of $M$, written $M^*$, to be the rank $m - r$ matroid given by*

$$\varphi(x_1, \ldots, x_{m-r}) = \operatorname{sign}(x_1, \ldots, x_{m-r}, x'_1, \ldots, x'_r) \cdot \varphi(x'_1, \ldots, x'_r)$$

*where $\{x'_1, \ldots, x'_r\} = E \backslash \{x_1, \ldots, x_{m-r}\}$. This is independent of the ordering of $x'_1, \ldots, x'_r$ that you choose.*

With this notion of dual it is easy to see that the double dual is the original matroid. With other notions it is a theorem.

## References

[BB16] M. Baker and N. Bowler, *Matroids over hyperfields*, arXiv:1601.01204