TROPICAL SCHEME THEORY

6. Bend relations and congruences, affine tropical schemes

We begin by reviewing set-theoretic tropicalization.

Definition 6.1. *A valuation on a ring R is a map v from R to an idempotent semiring S satisfying:*

1. $v(1_R) = 1_S$, $v(0_R) = 0_S$,
2. $v(-1_R) = 1_S$,
3. $v(ab) = v(a)v(b)$ $\forall a, b \in R$, *and*
4. $v(a) + v(b) = v(a+b) + v(a) + v(b)$ $\forall a, b \in R$.

Remark 6.2. Classically, v takes values in a totally ordered semiring. Under this assumption, condition (2) is redundant and condition (4) is equivalent to the statement that $v(a+b) \geq \min\{v(a), v(b)\}$, with equality if $v(a) \neq v(b)$.

Example 6.3. Let R be any domain. We can define a valuation $v : R \to \mathbb{B}$ such that $v(a) = 0_{\mathbb{B}}$ if $a = 0$ and $v(a) = 1_{\mathbb{B}}$ if $a \neq 0$. We call this the trivial valuation.

Example 6.4. Let R be $\mathbb{C}[t]$ or $\mathbb{C}[\![t]\!]$. Define $v : R \to \mathbb{T}_{\mathbb{Z}} = (\mathbb{Z} \cup \{\infty\}, \min, +)$ by $v(f) = \sup\{n \mid t^n \text{ divides } f\}$. Valuations whose image is a subset of $\mathbb{T}_{\mathbb{Z}}$ are called discrete valuations.

Example 6.5. Let $R = \mathbb{Z}$, p a prime. The function $v_p : \mathbb{Z} \to \mathbb{T}_{\mathbb{Z}}$ defined by $v_p(a) = \sup\{n \mid p^n \text{ divides } a\}$ is called the p-adic valuation.

Example 6.6. Let R be any ring, and let S be the semiring of ideals in R. The function $v : R \to S$ defined by $v(a) = \langle a \rangle$ is a valuation.

Question 6.7 (Sam)**.** *Does the map to the semiring of functions on the Berkovich analytification factor through the semiring of ideals?*

Definition 6.8. *Let $v : k \to \mathbb{T}$ be a valued field. The tropicalization is the map* $\mathrm{trop} : \mathbb{A}^n_k \to \mathbb{T}^n$ *sending a point $(a_1, \ldots, a_n)$ to $(v(a_1), \ldots, v(a_n))$.*

Question 6.9. *Given $f \in k[x_1, \ldots, x_n]$, what is $\mathrm{trop}(V(f))$?*

Example 6.10. Let $f(x) = x^2 - 1$. Then $V(f) = \{\pm 1\}$, and $\mathrm{trop}(V(f)) = \{0\}$. Consider the image of the graph of $f(x)$ under the tropicalization map, namely the points $(v(x), v(x^2-1))$ in $\mathbb{R}^2$. This is shown in Figure 1. Note that $v(x^2-1) \geq \min\{2v(x), 0\}$, with equality if $2v(x) \neq 0$. Observe that $\mathrm{trop}(V(f))$ is the set of points where $\min\{2v(x), 0\}$ is achieved at least twice.

Date: September 26, 2017, *Speaker*: David Jensen, *Scribe*: Netanel Friedenberg.

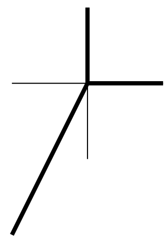


FIGURE 1.

The set-theoretic tropicalization of $V(f(x))$ in Example 6.10 is a single point. But we would like to realize it as a point with multiplicity two, since the original variety $V(f(x))$ has degree two. It is natural to ask if we can endow the image of the tropicalization with more algebraic or scheme-theoretic structure. Specifically, we want to find a semiring S such that $\operatorname{Hom}(S, \mathbb{T}) = \operatorname{trop}(V(f))$. Moreover, we want to think of S as the coordinate ring of the tropical variety in affine n-space, so S should be a quotient of $\mathbb{T}[x_1, \dots, x_n]$ by a congruence.

Definition 6.11. *Let S be an idempotent semiring and let $f \in S[x_1, \dots, x_n]$. Given a monomial $m \in \operatorname{supp}(f)$, denote by $f_{\widehat{m}}$ be the polynomial obtained from f after deleting the m-th term of f. The <u>bend relations</u> of f is the set of relations $\{f \sim f_{\widehat{m}}\}_{m \in \operatorname{supp}(f)}$. We write $\mathcal{B}(f)$ for the S-module congruence generated by the bend relations.*
If $J \subset S[x_1, \dots, x_n]$ is an ideal, we write $\mathcal{B}(J)$ for the S-module congruence generated by the bend relations of all $f \in J$.

: Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal generated by some $f_1, \dots, f_m$. Then in general $\mathcal{B}(\operatorname{trop}(I)) \neq \mathcal{B}(\operatorname{trop}(f_1), \dots, \operatorname{trop}(f_m))$, even in the case when $m = 1$. This can be seen in the following example.

Example 6.12. Consider the principal ideal I generated by $f = x^2 + xy + y^2$. We claim that $\mathcal{B}(\operatorname{trop}(f)) \subsetneq \mathcal{B}(\operatorname{trop}(I))$. Indeed, observe that I contains $f \cdot (x - y) = x^3 - y^3$ and thus $\operatorname{trop}(x^3 - y^3) = x^3 + y^3$ is in $\operatorname{trop}(I)$. Hence $\mathcal{B}(\operatorname{trop}(I))$ contains the relations $x^3 \sim x^3 + y^3$, $y^3 \sim x^3 + y^3$ and $x^3 \sim y^3$. But the relation $x^3 \sim y^3$ cannot be in $\mathcal{B}(\operatorname{trop}(f))$ since the generators of $\mathcal{B}(\operatorname{trop}(f))$ are of the form $g \sim h$ where g, h have at least two monomials.

Proposition 6.13. *Let S be a totally ordered idempotent semiring, and let $f \in S[x_1, \dots, x_n]$. An S-module homomorphism $p : S[x_1, \dots, x_n] \to S$ factors through $S[x_1, \dots, x_n]/\mathcal{B}(f)$ if and only if either $p(f) = 0_S$ or the minimum of the terms of $p(f)$ occurs at least twice.*

Proof. By definition, p factors through the quotient $S[x_1, \dots, x_n]/\mathcal{B}(f)$ if and only if $p(f) = p(f_{\widehat{m}})$ for all $m \in \operatorname{supp}(f)$. This occurs if and only if either $\operatorname{supp}(f) = \emptyset$ (i.e. $f = 0$) or no monomial is sent to something strictly smaller than all of the others. $\square$

Definition 6.14. *If S is a semiring and $(M, +)$ is a monoid, then we denote by $S[M]$ the monoid algebra $S[M] = \{\sum a_i x^{m_i} | a_i \in S, m_i \in M\}$ with $x^m \cdot x^{m'} = x^{m+m'}$.*

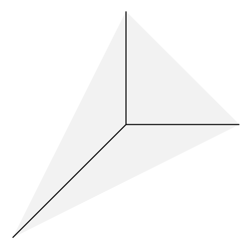
Example 6.15. The monoid algebra $k[\mathbb{N}^n]$ is the polynomial ring $k[x_1, \ldots, x_n]$. A monomial $x_1^{a_1} \cdots x_n^{a_n}$ corresponds to $x^{\vec{a}}$.

Example 6.16. The monoid algebra $k[\mathbb{Z}^n]$ is the Laurent polynomial ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

For any monoid algebra $S[M]$ and $I \in S[M]$ we can define $\mathcal{B}(I)$. This is useful since monoid algebras generalize coordinate rings of affine toric varieties.

Introduction to Toric Varieties.

A toric variety over a field k is determined by a fan in $\mathbb{R}^n$.



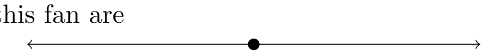
Given a cone $\sigma \subset \mathbb{R}^n$, its dual is the cone

$$\sigma^\vee = \{\vec{x} \in (\mathbb{R}^n)^\vee \mid \vec{x} \cdot \vec{y} \geq 0 \text{ for all } y \in \sigma\}.$$

The cone σ determines the affine variety $U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap \mathbb{Z}^n]$.

If $\tau \subset \sigma$ are cones in a fan Δ, then $\tau^\vee \supset \sigma^\vee$, so we have an inclusion $\sigma^\vee \cap \mathbb{Z}^n \hookrightarrow \tau^\vee \cap \mathbb{Z}^n$. This inclusion of monoids defines an inclusion of monoid algebras $k[\sigma^\vee \cap \mathbb{Z}^n] \hookrightarrow k[\tau^\vee \cap \mathbb{Z}^n]$, which in turn induces an open immersion $U_\tau \to U_\sigma$. Thus we can glue the affine toric varieties U_σ, by identifying U_τ with an open subvariety of U_σ, whenever τ is a face of σ. We denote the resulting variety by $X(\Delta)$. Note that the inclusion of $\tau = \{0\} \hookrightarrow \sigma$, for any $\sigma \in \Delta$ induces an inclusion of the torus $(k^\times)^n \hookrightarrow U_\sigma$. The action of $(k^\times)^n$ on itself extends to $X(\Delta)$, showing that $X(\Delta)$ a toric variety.

Example 6.17. Consider the fan Δ $\longleftarrow\!\bullet\!\longrightarrow$. The monoid algebras corresponding to each cone in this fan are



$k[x^{-1}]$ $\quad$ $k[x^{\pm 1}]$ $\quad$ $k[x]$

The inclusions $k[x] \hookrightarrow k[x^{\pm 1}] \hookleftarrow k[x^{-1}]$ of the coordinate rings of each U_σ, for $\sigma \in \Delta$ induce the inclusions $\mathbb{A}^1 \hookleftarrow k^\times \hookrightarrow \mathbb{A}^1$. Now we can see that variety $X(\Delta)$ associated to the fan Δ is $\mathbb{P}^1$.

Lemma 6.18. *Let $\varphi : M \to N$ be a map of monoids, and $f \in S[M]$. Then $\varphi_* \mathcal{B}(f) \subset \mathcal{B}(\varphi(f))$. Moreover, if φ is injective, then this containment is an equality.*

Proof. It suffices to show that any relation of the form $\varphi(f) \sim \varphi(f_{\widehat{m}})$ is implied by the relation $\varphi(f) \sim \varphi(f)_{\widehat{\varphi(m)}}$ since $\varphi_* \mathcal{B}(f)$ is generated as an S-module congruence by the image of the generators of $\mathcal{B}(\varphi(f))$. Let $g_0, \ldots, g_k$ be the terms of f corresponding to the monomials in $\varphi^{-1}(\varphi(m))$ with g_0 being the term with support m. Now

$$\begin{aligned}\varphi(f_{\widehat{m}}) &= \varphi(f)_{\widehat{\varphi(m)}} + \varphi(g_1 + \ldots + g_k) \\ &\sim \varphi(f) + \varphi(g_1 + \ldots + g_k) \\ &= \varphi(f),\end{aligned}$$

where the last equality holds by idempotence of addition of S.
If φ is injective the statement immediately follows since $\varphi(f_{\widehat{m}}) = \varphi(f)_{\widehat{\varphi(m)}}$. □

References

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