TROPICAL SCHEME THEORY

7. Tropical schemes

In this lecture we present the construction of tropical schemes.

Definition 7.1. *Let A be an idempotent semiring. The prime spectrum of A, denoted* $\operatorname{Spec} A$*, is the set of all prime ideals of A. We equip* $\operatorname{Spec} A$ *with the Zariski topology: The open sets are $\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \not\subset \mathfrak{p}\}$ for fixed ideals $\mathfrak{a}$; the closed sets are $V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{a} \subset \mathfrak{p}\}$ for fixed ideals $\mathfrak{a}$. A base for the topology is given by $D(f) = \{\mathfrak{p} | f \notin \mathfrak{p}\}$ for $f \in A$.*

Any A-module (A-algebra) M defines a sheaf $\widetilde{M}$ of A-modules (resp. A-algebras) that sends $D(f) \mapsto M_f = A_f \otimes M$ where A_f is the localization of A by f. In particular, for $M = A$ this gives us a sheaf of semirings, which is called the structure sheaf $\mathcal{O}_A$.

An affine tropical scheme is a pair $(X, \mathcal{O})$ such that X is a topological space and $\mathcal{O}$ is a sheaf of semirings (or $\mathbb{T}$-algebras) that is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_A)$ *for some A. A tropical scheme is a pair $(X, \mathcal{O})$ which is locally an affine tropical scheme.*

Remark 7.2. Classically, if k is a field and $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ an ideal, the zero locus of $\mathfrak{a}$ defines a subscheme of $\mathbb{A}^n_k$ with coordinate ring $k[x_1, \ldots, x_n]/a$. However, quotients by ideals are not well-behaved when we are working over a semiring.

The theory of open subschemes and gluing is identical for schemes and semiring schemes. However, defining closed subschemes is more subtle in the case of semirings schemes, in view of Remark 7.2.

We recall the definition of closed immersion for classical schemes:
A closed immersion is a morphism of schemes $\varphi : Y \to X$ such that (1)

- $\varphi(Y) \subset X$ is (topologically) a closed subspace of X,
- the induced map $\varphi : Y \to \varphi(Y)$ is a homeomorphism, and
- $\varphi^\# : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is surjective.

Equivalently, φ is a closed immersion (2) if

- φ is an affine morphism, and
- $\varphi^\# : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is surjective.

Remark 7.3. If X and Y are semiring schemes (1) and (2) are no longer equivalent. In fact, if φ is an affine morphism with $\varphi^\#$ surjective, then $\varphi(Y)$ is not necessarily topologically closed in X as shown in Example 7.4.

Example 7.4. Consider $\varphi : \operatorname{Spec} \mathbb{T} \to \operatorname{Spec} \mathbb{T}[x] = \mathbb{A}^1_{\mathbb{T}}$ given by a semiring homomorphism $\varphi^\# : \mathbb{T}[x] \twoheadrightarrow \mathbb{T}$ sending x to some $t \in \mathbb{R}$, i.e. $t \neq 0_{\mathbb{T}}$. Note that the ideal-theoretic kernel of $\varphi^\#$ is $(\varphi^\#)^{-1}(0_{\mathbb{T}}) = 0_{\mathbb{T}[x]}$, and it is contained in every prime

Date: October 3, 2017, *Speaker*: Kalina Mincheva, *Scribe*: Netanel Friedenberg.

ideal of $\mathbb{T}[x]$. So the set-theoretic image of φ is a point which is dense in $\operatorname{Spec}\mathbb{T}[x]$, i.e. it is the generic point of $\mathbb{A}^1_{\mathbb{T}}$.

In view of Remark 7.3 we define a closed immersion $\varphi : Y \to X$ of semiring schemes in analogy with (2).

Definition 7.5. *A morphism of tropical schemes $\varphi : X \to Y$ is a closed immersion if it is affine and $\varphi^{\#} : \mathcal{O}_X \to \varphi_*\mathcal{O}_Y$ is surjective.*

A closed subscheme is an equivalence class of of closed immersions where $\varphi : Y \to X$ and $\varphi' : Y' \to X$ are equivalent if there is an isomorphism $Y \cong Y'$ commuting with φ, φ'. We define a closed tropical subscheme in the same way.

Example 7.6. Let a A be a ring and $a \subset A$ an ideal. For $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/a$ the ring homomorphism $A \to A/a$ gives a closed immersion $Y \xrightarrow{\varphi} X$. In fact, all closed subschemes of X arise this way. Moreover, $\varphi_*\mathcal{O}_Y$ is quasi-coherent and $\varphi_*\mathcal{O}_Y \cong \widetilde{A/a}$.

The lack of bijective correspondence between the congruences and the ideals of a semiring motivates the following definition:

Definition 7.7. *A congruence sheaf $\mathcal{J}$ on X is a subsheaf (of $\mathcal{O}_X$-modules) of $\mathcal{O}_X \times \mathcal{O}_X$ such that $\mathcal{J}(U)$ is a congruence of $\mathcal{O}_X(U)$ for each open $U \subset X$. $\mathcal{J}$ is quasi-coherent if it is quasi-coherent when we regard it as a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X \times \mathcal{O}_X$.*

Proposition 7.8. *Let $X = \operatorname{Spec} A$ be an affine tropical scheme. Then taking global sections gives a 1-1 correspondence between congruences on A and quasi-coherent congruence sheaves on $\operatorname{Spec} A$.*

Proposition 7.9. *For a general tropical scheme there is a 1-1 correspondence between closed subschemes of X and quasi-coherent congruence sheaves on X.*

We now try to understand the prime ideals and congruences of $\mathbb{T}[x]$ as ways to understand the points of $\mathbb{T}$:

(1) For $\operatorname{Spec}\mathbb{T}[x] = \mathbb{A}^1_{\mathbb{T}}$ we have that

$$\operatorname{Hom}_{\mathrm{Sch}/\mathbb{T}}(\operatorname{Spec}\mathbb{T}, \mathbb{A}^1_{\mathbb{T}}) \cong \operatorname{Hom}_{\mathbb{T}\text{-alg}}(\mathbb{T}[x], \mathbb{T}) \cong \mathbb{T}.$$

For $\varphi \in \operatorname{Hom}_{\mathbb{T}\text{-alg}}(\mathbb{T}[x], \mathbb{T})$ with $\varphi(x) = a \neq 0_{\mathbb{T}}$, we have that the ideal-theoretic kernel of φ is $\varphi^{-1}(0_{\mathbb{T}}) = \{0_{\mathbb{T}}\}$. In order to differentiate between the points of $\mathbb{A}^1_{\mathbb{T}}$ we can consider the congruence-theoretic kernel $\ker\varphi = \varphi^{-1}(\Delta)$.

(2) We now look at the prime congruences of $\mathbb{T}[x]$:

- Consider the congruence-theoretic kernels of the evaluation maps $\mathbb{T}[x] \xrightarrow{\mathrm{ev}_t} \mathbb{T} : x \mapsto t$. It is easy to see that these are prime congruences. We will suggestively call them geometric primes. They correspond to points on $\mathbb{T}$.
- There are other prime congruences P such that for every $a \in \mathbb{T}\setminus\{0_{\mathbb{T}}\}$, $(a, 1_{\mathbb{T}}) \in \mathbb{T}[x]/P$. These correspond to primes of $\mathbb{B}[x]$ since $\mathbb{T}[x]/P \cong \mathbb{B}[x]/P_B$ for some prime congruence P_B of $\mathbb{B}[x]$.

(3) We can treat the elements of $\mathbb{T}[x]$ as functions rather than polynomials. To do that, we consider $\widetilde{\mathbb{T}} = \mathbb{T}[x]/\sim$ where $f \sim g$ if $f(t) = g(t)$ for all $t \in \mathbb{T}$. If $f \in \mathbb{B}[x]$ the image of f in $\widetilde{\mathbb{T}}$ is determined by the Newton polytope of f. If we allow non-trivial coefficients, in $\widetilde{\mathbb{T}}$ we remember the data of a regular subdivision of the Newton polytope of f together with the heights of vertices in the subdivision. There are two advantages in working with $\widetilde{\mathbb{T}}$.

- The semiring $\widetilde{\mathbb{T}}$ is cancellative.
- Every element of $\widetilde{\mathbb{T}}$ can be written as a product $a\prod_i f_i$ with each f_i linear and of one of the forms $f_t = 1_{\mathbb{T}} + t^{-1}x$ for $t \in \mathbb{T}\setminus\{0_{\mathbb{T}}\}$ or $f_{0_{\mathbb{T}}} = x$. The prime ideals of $\widetilde{\mathbb{T}}$ are characterized by the following proposition.

Proposition 7.10. *Let K be an interval of $\mathbb{T}$, not necessarily closed or open, and let $I_K \setminus \{0_{\mathbb{T}}\}$ be the set of functions that have a bend on K. Then*

$$|\operatorname{Spec}\widetilde{\mathbb{T}}| = \{I_K \mid K \subset \mathbb{T} \text{ is an interval}\}.$$

Moreover, finitely generated primes correspond to closed intervals and principal primes correspond to points.

Proof. We first show that if K is an interval then $I_K = \{0_{\mathbb{T}}\} \cup \{$functions that have a bend in $K\}$. If $f \in \widetilde{\mathbb{T}}$ bends at $t \in K$ then f_t divides f and so $f \in I_K$. For the other inclusion, since for $t \in K$, f_t bends at the point $t \in K$ it suffices to show that I_K is an ideal of $\widetilde{\mathbb{T}}$. Consider $f, g \in I_K$ and $f, g \neq 0_{\widetilde{\mathbb{T}}}$ (the case where either one of them is $0_{\widetilde{\mathbb{T}}}$ is trivial). If f bends at $t_1 \in K$ and g bends at $t_2 \in K$ with $t_1 \leq t_2$. Then $f + g$ bends on the interval $[t_1, t_2] \subset K$, so $f + g \in I_K$. If $f \in I_K$ and $g \in \widetilde{\mathbb{T}}$ then gf is either $0_{\widetilde{\mathbb{T}}}$ or bends everywhere that f does, so $gf \in I_K$.

Now we show that $I_K = I_{\operatorname{Conv}(K)}$, where $\operatorname{Conv}(K)$ is the convex hull of K. Clearly $I_K \subset I_{\operatorname{Conv}(K)}$. For the other inclusion, say $t_1, t_2 \in K$ with $t_1 < t_2$ and $r \in [t_1, t_2)$. If $t_2 \neq 0_{\mathbb{T}}$ then $f_r = f_{t_1} + t_2 r^{-1} f_{t_2} \in I_K$ and if $t_2 = 0_{\mathbb{T}}$ then $f_r = f_{t_1} + r^{-1} f_{t_2} \in I_K$. Thus $I_{\operatorname{Conv}(K)} \subset I_K$.

To see that I_K is prime, note that we have $I_K = I_{\operatorname{Conv}(K)} = \{0_{\mathbb{T}}\} \cup \{$functions that have a bend in $\operatorname{Conv}(K)\}$. But if f and g don't bend on $\operatorname{Conv}(K)$ then fg also does not bend on $\operatorname{Conv}(K)$.

Since every $f \in \widetilde{\mathbb{T}}\setminus\{0_{\widetilde{\mathbb{T}}}\}$ can be written as $a\prod_{i=1}^n f_{t_i}$ with a a unit, we see that every prime ideal $P \subset \widetilde{\mathbb{T}}$ is generated by those f_t which are in P. Thus for any prime ideal P we have $P = (f_t \mid f_t \in P) = I_{\{t \mid f_t \in P\}}$.

If P is a prime of $\widetilde{\mathbb{T}}$ then every element of P has to be divisible by some f_t. Note that $t_1 r^{-1} f_{t_1} + f_{t_2} = f_r$ for $r \in [t_1, t_2]$. So $P = I_K$ for some interval K. □

The above proposition may indicate that there are too many prime ideals.

We now define scheme-theoretic tropicalization.

Affine case: Let k be a valued field and let $X = \operatorname{Spec} k[x_1, \ldots, x_n]/I$. We associate to X the tropical scheme $X_{\mathbb{T}} = \operatorname{Spec}\mathbb{T}[x_1, \ldots, x_n]/\mathcal{B}(v(I))$, called the scheme-theoretic tropicalization of X. The set of $\mathbb{T}$-points of $X_{\mathbb{T}}$ is $\operatorname{trop} X$ the set-theoretic tropicalization of X, which is also equal to $\mathrm{Hom}_{\mathbb{T}\text{-alg}}(\mathbb{T}[x_1,\dots,x_n]/\mathcal{B}(v(I)),\mathbb{T})$.

We know that $\dim X = \dim(\operatorname{trop} X)$. But we need to make sense of dimension of tropical schemes, which we will define in terms of the Hilbert polynomial, and then we can see that this also equals $\dim X_{\mathbb{T}}$. This will also be equal to the Krull dimension of $\mathbb{T}[x_1,\dots,x_n]/\mathcal{B}(v(I))$ minus one.

We recall the classical set-up. Consider the graded algebra $A = k[x_0,\dots,x_n]$. $\operatorname{Proj} A = \mathbb{P}^n_k$. A subscheme $Z \subset \mathbb{P}^n_k$ is defined by a homogeneous ideal I. The Hilbert function of A/I is $d \mapsto \dim_k(A/I)_d = \dim_k(A/I)^\vee_d$. The corresponding Hilbert series is $\mathrm{HS}(t) = \sum_{d=0}^{\infty} \mathrm{hilb}(d)t^d$. For $d \gg 0$ $\mathrm{hilb}(d)$ is a polynomial in d, called the Hilbert polynomial of Z, denoted HP_I.

Example 7.11. We compute the Hilbert function $\mathrm{HS}(t)$

- $A = k[x]$. $\mathrm{HS}(t) = 1 + t + t^2 + \cdots = \frac{1}{1-t}$.
- $A = k[x,y]$. $\mathrm{HS}(t) = 1 + 2t + 3t^2 + \cdots = \frac{1}{(1-t)^2}$.

In order to build the corresponding theory of Hilbert functions, series, and polynomials in the tropical case, we need to have a notion of the dimension of a $\mathbb{T}$-module. The following definition is proposed in [MZ08].

Definition 7.12. *Let L be a $\mathbb{T}$-module.*

(1) A collection $v_1,\dots,v_k \in L$ are called linearly dependent if any finite linear combination of the v_is can be written as a linear combination of a proper subset of $\{v_1,\dots,v_k\}$. Otherwise, we say that $v_1,\dots,v_k$ are linearly independent.

(2) $\dim_{\mathbb{T}} L$ is the largest number d such that there is a set of d linearly independent elements of L.

Lemma 7.13. *Let L be a tropical linear space of rank r, then $\dim_{\mathbb{T}} L = r$.*

Definition 7.14. *Given a homogeneous congruence C on $\mathbb{T}[x_0,\dots,x_n]$ the Hilbert function of $\mathbb{C}$ is the map $d \mapsto \dim_{\mathbb{T}}(\mathbb{T}[x_0,\dots,x_n]/C)^\vee_d$.*

If $C = \mathcal{B}(v(I))$ for an ideal $I \subset K[x]$, then we can think of $(\mathbb{T}[x]/C)^\vee_d$ a a tropical linear space which is the tropicalization of $(K[x]/I)_d$, regarding the latter as a vector space. We provide the details in the following lecture.

Remark 7.15. Two homogeneous congruences define the same projective scheme if they coincide in sufficiently large degree.

Definition 7.16. *The saturation C^{sat} of a homogeneous congruence is the maximal congruence that agrees with C in sufficiently large degree.*

Definition 7.17. *The Hilbert function of a subscheme $Z \subset \mathbb{P}^n_{\mathbb{T}}$ is the Hilbert function of C^{sat} for any congruence C defining Z.*

Theorem 7.18. *Let $v : k \to \mathbb{T}$ be a valuation, and $I \subset k[x_0,\dots,x_n]$ an ideal. Let Z_k be the closed subscheme of $\mathbb{P}^n_k$ defined by I, and let $Z_{\mathbb{T}}$ be the scheme-theoretic* *tropicalization of* Z_k*. Then* $\mathrm{hilb}_I = \mathrm{hilb}_{\mathcal{B}(v(I))}$*, and so the Hilbert polynomials of* Z_k *and* $Z_{\mathbb{T}}$ *are equal.*

Classically the dimension of a scheme Z_k with defining ideal I is the degree of the Hilbert polynomial. The dimension of a tropical scheme is defined in the same way. In particular, with the notation from Theorem 7.13 $\dim Z_k = \dim Z_{\mathbb{T}} = \deg \mathrm{HP}_I = \deg \mathrm{HP}_{\mathcal{B}(v(I))}$.

Proof. First note that $B(v(\_))$ commutes with restrictions to degree d graded piece. Then note that $(\mathbb{T}[x_0, \ldots, x_n]/\mathcal{B}(v(I)))_d = \mathbb{T}[x_0, \ldots, x_n]_d/\mathcal{B}(v(I_d))$. The dual of the right hands side is the tropical linear space, which is the tropicalization of $k[x_0, \ldots, x_n]_d/I_d$. Since the tropicalization of a subspace of dimension r has rank r, then by Lemma 7.13 the tropical linear space has dimension r as well. $\square$

Theorem 7.19 (Theorem 7.2.1 [Mi16])**.** *The Krull dimension of* $\mathbb{T}[x_1, \ldots, x_n]/\mathcal{B}(v(I))$ *is the Krull dimension of* $k[x_1, \ldots, x_n]/I$ *plus one.*

References

[MZ08] G. Mikhalkin, I. Zharkov, *Tropical curves, their Jacobians and theta functions*, Curves and abelian varieties 465 (2008): 203-230.

[Mi16] K. Mincheva, *Prime congruences and tropical geometry*, PhD thesis (2016)