TROPICAL SCHEME THEORY

8. Universal tropicalization

Classically, a valuation is a map $v : k\backslash\{0\} \to \mathbb{R}$ where k is a field. However, we can extend the notion of valuation to $v : k \to \mathbb{R} \cup \{\infty\}$, or even more generally when R a ring, we can consider $v : R \to (\mathbb{R} \cup \{\infty\}, \min, +) = \mathbb{T}$ such that

- $v(ab) = v(a)v(b)$,
- $v(1) = 1_{\mathbb{T}}$,
- $v(0) = 0_{\mathbb{T}}$, and
- $v(a + b) \geq v(a) + v(b)$.

Let S be an arbitrary idempotent semiring. In particular, S is partially ordered, but not necessarily totally ordered. We can define an even more general valuation $v : R \to S$ by replacing the inequality $v(a + b) \geq v(a) + v(b)$ with the condition

$$v(a + b) + v(a) + v(b) = v(a) + v(b).$$

We also want $v(0_R) = 0_S$ and $v(\pm 1_R) = 1_S$. If we have an ordered group A, then to get back the usual notion of valuation, build S as $(A \cup \{\infty\}, +_S = \min, \cdot_S = +_A)$.

Note that if S is totally ordered then multiplicativity and $v(1) = 1$ already imply $v(-1) = 1$. To see this note that we have $v(-1)^2 = 1 = v(1)^2$. Moreover, 1 is the unique square root of 1 since if $a < 1$ then $a^2 \leq a < 1$ and if $a > 1$ then $a^2 \geq a > 1$.

Lemma 8.1. *Let $v : R \to S$ be a valuation. For all $a, b \in R$ we have*

$$v(a + b) + v(a) + v(b) = v(a + b) + v(a).$$

Proof. Let $x = a + b$ and $y = -a$. Then

$$\begin{aligned} v(b) + v(a + b) + v(a) &= v(x + y) + v(x) + v(y) \\ &= v(x) + v(y) = v(a + b) + v(-a). \end{aligned}$$

□

We can now define the universal valuation on a fixed ring R.

Definition 8.2. *Let R be a ring and let*

$$S^R_{\text{univ}} := \frac{\mathbb{B}[x_a \,|\, a \in R]}{\left\langle \begin{array}{c} x_0 \sim 0,\ x_1 \sim x_{-1} \sim 1, \\ x_{ab} \sim x_a x_b,\ \textit{and} \\ \mathcal{B}(x_a + x_b + x_{a+b})\ \textit{for all}\ a, b \in R \end{array} \right\rangle}$$

where $\mathcal{B}(f)$ denotes the set of bend relations of a polynomial f. The <u>*universal valuation*</u> *on R is the valuation $v : R \to S^R_{\text{univ}}$ given by $a \mapsto x_a$.*

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For the rest of the lecture, all monoids are commutative and are written multiplicatively.

Let R be a ring and A an R-algebra. We construct the monoid algebra $R[A]$, where we think of the elements of A as formal variables and we replace the addition on A by a formal addition operation. We have an evaluation map $\mathrm{ev} : R[A] \twoheadrightarrow A$ given by

$$\sum r_i x_{a_i} \mapsto \sum r_i a_i.$$

Proposition 8.3. *Given a monoid M and an R-algebra homomorphism $f : R[M] \to A$, there exists a unique monoid homomorphism $g : M \to A$ such that*

$$\begin{array}{ccc} R[M] & \xrightarrow{f} & A \\ {\scriptstyle R[g]}\downarrow & \nearrow_{\mathrm{ev}} & \\ R[A] & & \end{array}$$

commutes.

The evaluation map $R[A] \overset{\mathrm{ev}}{\twoheadrightarrow} A$ induces a closed embedding

$$\operatorname{Spec} A \hookrightarrow \operatorname{Spec} R[A]. \tag{1}$$

Thus we can think of $\operatorname{Spec} R[A]$ as an "infinite type toric variety". Thus in view of Proposition 8.3 we can regard (1) as a universal morphism for all maps from $\operatorname{Spec} A$ to toric varieties over R.

Proposition 8.4. *The kernel of* $\mathrm{ev} : R[A] \twoheadrightarrow A$ *is generated as a $\mathbb{Z}$-module by*

$$(*) \begin{cases} (1)\ r x_a - x_{ra} & \text{for all } r \in R \text{ and } a \in A \\ (2)\ x_a + x_b + x_c & \text{for all } a, b, c \in A \text{ such that } a + b + c = 0. \end{cases}$$

Let R be a ring and A an R-algebra. If we have a valuation $v : R \to S$ and A is a domain we can tropicalize $\operatorname{Spec} R[A]$. We denote by X the scheme-theoretic tropicalization of $\operatorname{Spec} R[A]$. X is a defined as an affine tropical scheme by the set of bend relations which we will denote by $\mathcal{B}(X)$.

Proposition 8.5. *The elements in (*) from a <u>strong tropical basis</u>. That is, they generate $\mathcal{B}(X)$ as a congruence.*

Definition 8.6. *If T is an S-algebra then a valuation* $\mathrm{val} : A \to T$ *is <u>compatible</u> with $v : R \to S$ if the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{val}} & T \\ \uparrow & & \uparrow \\ R & \xrightarrow[v]{} & S \end{array}$$

commutes.

We have a covariant functor $\mathrm{S-alg} \to \mathrm{Sets}$ given by $T \mapsto \{$valuations $A \to T$ compatible with $v\}$.

Theorem 8.7. *Let X be the scheme-theoretic tropicalization of* $\operatorname{Spec} R[A]$*. Then X represents the above functor, i.e. X is the moduli space of valuations on A. In particular, If $R = k$ and $S = \mathbb{T}$ so that we are dealing with a classical (rank 1) valuation on a field,* $\operatorname{Hom}(\operatorname{Spec} \mathbb{T}, X)$ *is the same as the underlying set of the Berkovich analytification of* $\operatorname{Spec} A$*.*

The universal property of $R[A]$ given in Proposition 8.3 together with functoriality of scheme-theoretic tropicalization with respect to toric morphisms shows that X maps (scheme theoretically) to any other embedded tropicalization of $\operatorname{Spec} A$. So we see that trivially, X is the limit of all tropicalizations of $\operatorname{Spec} A$. We can get a finer result.

Theorem 8.8. *X is isomorphic to the limit of the tropicalizations* $\operatorname{Trop}(\operatorname{Spec} A \to \mathbb{A}^n_R)$ *over all embeddings of* $\operatorname{Spec} A$ *into affine space over R.*