TROPICAL SCHEME THEORY

9. Tropical schemes, ideals of tropical (Laurent) polynomial semirings, and valuated matroids

We will discuss the relation between the Tropical schemes, ideals of tropical (Laurent) polynomial semirings, and valuated matroids.

Theorem 9.1. *Let k be a field and let $v : k \to \mathbb{T}$ be a valuation. For $Y \subset (k^\times)^n$ a subscheme defined by an ideal $I \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, any of the following objects determine the others.*

1. *The congruence $\mathcal{B}(\operatorname{trop} I) = \mathcal{B}(v(I))$ on $\mathbb{T}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,*
2. *The ideal $\operatorname{trop} I = \{\operatorname{trop} f \mid f \in I\}$, and*
3. *The set (tower) of valuated matroids of the vector space I_d^h, where $I^h \subset k[x_0, \ldots, x_n]$ is the homogenization of I and I_d^h is the degree d part of I^h.*

Homogenization:

- The homogenization of a polynomial $f = \sum c_u x^u \in k[x_1, \ldots, x_n]$ is
$$\tilde{f} = \sum c_u x^u \cdot x_0^{-|u| + \deg(f)},$$
where $|u| = u_1 + \cdots + u_n$ and $\deg f = \max\{|u| \mid c_u \neq 0\}$. For example, the homogenization of $f = x_1^2 x_2^3 + x_3$ is $\tilde{f} = x_1^2 x_2^3 + x_3 x_0^4$.
- The homogenization of a tropical polynomial $F = \sum a_u x^u = \min(a_u + x \cdot u)$ is
$$\tilde{F} = \min(a_u + x \cdot u + (\deg(F) - |u|)x_0).$$
- The homogenization of an ideal $I \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is
$$I^h = \{\tilde{f} \mid f \in I \cap k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\},$$
which is an ideal of $k[x_0, x_1, \ldots, x_n]$.
- The homogenization of a relation $F \sim G$ with $F, G \in \mathbb{T}[x_1, \ldots, x_n]$ (i.e. $\operatorname{supp}(F), \operatorname{supp}(G) \subset \mathbb{N}^n$) with $\deg F \geq \deg G$ is
$$\tilde{F} \sim \tilde{G} + (\deg(F) - \deg(G))x_0.$$
- The homogenization of a congruence $J \subset \mathbb{T}[\underline{x}] \times \mathbb{T}[\underline{x}]$ is the congruence
$$J^h = \left\langle \widetilde{F \sim G} \middle| (F, G) \in J \right\rangle.$$

Proposition 9.2. *Let $I \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be an ideal. Then $\mathcal{B}(\operatorname{trop}(I^h)) = (\mathcal{B}(\operatorname{trop} I))^h$.*

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Before we proceed with the proof of Theorem 9.1 we recall the definition of a tropical linear space:

Definition 9.3. *Let M be a matroid on a finite set $E \cong \{0, 1, \ldots, n\}$. The tropical linear space* $\operatorname{trop}(M)$ *is the set of vectors $(w_0, \ldots, w_n) \in \mathbb{R}^{n+1}$ such that for any circuit C of M the minimum of the w_is is attained at least twice as i ranges over C.*

Proof Theorem 9.1, part 1. Here we show that (1) and (2) determine each other. (The second part of the proof, showing that (2) and (3) determine each other, is postponed until the discussion of valuated matroids and tropical linear spaces, below.) It is easy to see that $\operatorname{trop} I$ determines $\mathcal{B}(\operatorname{trop} I)$. Conversely, by Proposition 9.2 we have that $\mathcal{B}(\operatorname{trop} I)$ determines $\mathcal{B}(\operatorname{trop} I^h)$. We can regard any homogeneous polynomial $f \in I_d$ as a linear form on $\mathbb{A}^{\binom{n+d}{d}}$ whose coordinates are indexed by monomials of degree d in $k[x_1, \ldots, x_n]$. Similarly, we can regard a tropical polynomial F of degree d as a tropical linear form on $\mathbb{T}^{\binom{n+d}{d}}$. If L_d is the subspace of $\mathbb{A}^{\binom{n+d}{d}}$ where the linear forms l_f vanish for all $f \in I_d$, then

$$\begin{aligned}
\operatorname{trop}(L_d) = l_d :&= \left\{ z \in \mathbb{T}^{\binom{n+d}{d}} \,\middle|\, \operatorname{trop} l_f(z) \text{ attains its minimum at least twice} \right\} \\
&= \left\{ z \in \mathbb{T}^{\binom{n+d}{d}} \,\middle|\, \operatorname{trop} l_f(z) = \operatorname{trop} l_{f_{\hat{u}}}(z) (\forall f \in I_d^h)(\forall u \in \operatorname{supp}(f)) \right\} \\
&= \left\{ z \in \mathbb{T}^{\binom{n+d}{d}} \,\middle|\, l_{\operatorname{trop} f}(z) = l_{\operatorname{trop} f_{\hat{u}}}(z) (\forall f \in I_d^h)(\forall u \in \operatorname{supp}(f)) \right\} \\
&= \left\{ z \in \mathbb{T}^{\binom{n+d}{d}} \,\middle|\, l_f(z) = l_g(z) \left(\forall (f, g) \in \mathcal{B}(\operatorname{trop} I_d^h)\right) \right\}.
\end{aligned}$$

It follows that $\mathcal{B}(\operatorname{trop} I^h)$ determines the tower $\{l_d\}_{d \geq 0}$. But l_d also determines its dual linear space $l_d^\perp = \operatorname{trop}(L_d^\perp)$. Note that a vector lies in $L_d^\perp$ if and only if it is a coefficient vector for a polynomial in I_d^h. Thus the tropical linear space $l_d^\perp$ is the same as $\operatorname{trop}(I_d^h)$ (the degree d part of $\operatorname{trop}(I^h)$). This implies that $\mathcal{B}(\operatorname{trop} I^h)$ determines $\operatorname{trop} I^h$. Since we can recover I as $I^h|_{x_0=1}$, this determines $\operatorname{trop} I$. ☐

We proceed to introduce valuated matroids and explain their connection to tropical linear spaces.

Valuated matroids and tropical linear spaces

Let E a finite set and $r \in \mathbb{N}$. Let $\binom{E}{r}$ denote the collection of subsets of E of size r. A valuated matroid $\mathcal{M}$ on E of rank r is a function $p : \binom{E}{r} \to \mathbb{R} \cup \{\infty\}$, called the basis valuation function, such that

(1) $\exists B \in \binom{E}{r}$ such that $p(B) \neq \infty$ and
(2) for any $B, B' \in \binom{E}{r}$, for all $u \in B \backslash B'$ there exists $v \in B' \backslash B$ such that $p(B) - p(B') \geq p((B \backslash \{u\}) \cup \{v\}) + p((B' \backslash \{v\}) \cup \{u\})$.

The support of p, defined to be $\operatorname{supp}(p) = \{B \in \binom{E}{r} \mid p(B) \neq \infty\}$, is the collection of bases of a rank r matroid on E. We call this matroid "the underlying matroid of $\mathcal{M}$" and denote it by $\underline{\mathcal{M}}$.

Denote by M_d the set of all monomials of degree d. If f is a homogeneous polynomial of degree d in $k[x_0, \ldots, x_n]$, then we can think of f as a linear form l_f on the vector space V_d with basis M_d.

Let I_d be the degree d part of a homogeneous ideal $I \subset k[x_0, \ldots, x_n]$. Then consider $L_d = \{y \in V_d \mid l_f(y) = 0 (\forall f \in I_d)\} \subset V_d$. We have a pairing $\langle -, - \rangle : k[x_0, \ldots, x_n]_d \times V_d \to k$ given by $\langle f, y \rangle = l_f(y)$. Then the space L_d is the annihilator of I_d.

Let $\dim L_d =: r_d$. Another way to view L_d is as a point on the Grassmannian $Gr(r_d, V_d) \subset \mathbb{P}^N$, where $N = \binom{|M_d|}{r_d} - 1$. The valuated matroid $\mathcal{M}(I_d)$ of I_d is the function $p_d : \binom{M_d}{r_d} \to \mathbb{R} \cup \{\infty\}$ defined by letting $p_d(B)$ be the valuation of the Plücker coordinate of L_d indexed by B.

The (matroid-theoretic) vectors of $\mathcal{M}(I_d)$ are tropical polynomials. Vectors of minimal support are circuits of $\mathcal{M}(I_d)$. For example $F = \min(a_1 + xu_1, a_2 + xu_2)$, from which we get the vector $\{u_1, u_2\}$ of the underlying matroid.

Proof of Theorem 9.1, part 2. We can now show that (2) and (3), i.e., $\operatorname{trop}(I)$ and $\{\mathcal{M}(I_d)\}_{d \geq 0}$ determine each other. As discussed above the elements of $\operatorname{trop}(I)_d$ are the vectors of the valuated matroid $M(I_d)$, so $\operatorname{trop}(I)$ determines and is determined by the set of valuated matroids $\{\mathcal{M}(I_d)\}_{d \geq 0}$. □

Theorem 9.4. *Let $Y \subset (k^\times)^n$ be the subscheme defined by an ideal $I \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the multiplicities of the maximal cells of* $\operatorname{trop}(Y)$ *can be recovered from* $\mathcal{B}(\operatorname{trop} I)$.

This can be thought of as the tropical Hilbert-Chow morphism.

Multiplicities in tropical geometry

We recall the definition of multiplicities for the maximal cells of a tropical variety. Let Y be a subvariety of $(k^\times)^n$ of dimension d. Let $v : k^\times \to \Gamma$ be a valuation which admits a splitting $\Gamma \to k^\times$, i.e. a group homomorphism $w \mapsto t^w$ such that $v(t^w) = w$. First recall that $\operatorname{trop}(Y)$ is a polyhedral complex of pure dimension d.

For $w \in \mathbb{R}^n$ and $f = \sum c_u x^u$ the initial form of f with respect to w is

$$\operatorname{in}_w f = \sum_{\operatorname{val}(c_u) + w \cdot u = \operatorname{trop}(f)(w)} t^{-\operatorname{val}(c_u)} c_u x_u \in \bar{k}[x_1, \ldots, x_n],$$

where $\bar{k}$ is the residue field. We can similarly define the initial ideal $\operatorname{in}_w I = \langle \operatorname{in}_w f \mid f \in I \rangle$.

The multiplicity of w is defined to be

$$\operatorname{mult}(w) = \sum_{\substack{P \text{ a minimal associated} \\ \text{prime of } \operatorname{in}_w I}} \operatorname{mult}(P, \operatorname{in}_w I),$$

where $\operatorname{mult}(P, \operatorname{in}_w I)$ is the multiplicity of P in a primary decomposition of $\operatorname{in}_w I$.

After a suitable monomial change of coordinates on $(k^\times)^n$, we can show that $\operatorname{in}_w I$ is generated by polynomials in $x_{d+1}, \ldots, x_n$. Then $\operatorname{mult}(w) = \dim_{\bar{k}} \left(\frac{\bar{k}[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}]}{\operatorname{in}_w I \cap \bar{k}[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}]} \right)$. Indeed, one observes that the multiplicities do not change when we pass to $\bar{k}[x_{d+1}^{\pm}, \ldots, x_n^{\pm}]$ and then the problem is reduced to computing primary decomposition of the zero ideal in the Artinian $\bar{k}$-algebra $\frac{\bar{k}[x_{d+1}^{\pm 1},\dots,x_n^{\pm 1}]}{\operatorname{in}_w I\cap \bar{k}[x_{d+1}^{\pm 1},\dots,x_n^{\pm 1}]}$ (cf. Lemma 3.4.7 in [MS]).

To define the multiplicity of a maximal cell σ, pick w in the relative interior of σ. Then $\operatorname{mult}(\sigma) = \operatorname{mult}(w)$.

In order to recover the multiplicities from the bend relations, we need to extend Gröbner theory to congruences.

Let $F = \sum_u a_u x^u = \min(a_u + x\cdot u) \in \mathbb{T}[x_0,\dots,x_n]$. For $w \in \mathbb{R}^{n+1}$ the we define the initial form of F with respect to w to be $\operatorname{in}_w F := \min_{a_u + w\cdot u = F(w)} x\cdot u \in \mathbb{B}[x_0,\dots,x_n]$. For $G = \min(b_u + x\cdot u) \in \mathbb{T}[x_0,\dots,x_n]$ we consider the relation $F \sim G$. The initial form of $F \sim G$ is defined to be $\operatorname{in}_w(F\sim G) := \begin{cases} (\operatorname{in}_w F \sim \operatorname{in}_w G) & \text{if } F(w) = G(w) \\ (0 \sim \operatorname{in}_w G) & \text{if } F(w) > G(w) \end{cases}$.

Definition 9.5. *If C is a congruence then* $\operatorname{in}_w C = \langle \operatorname{in}_w(F\sim G) \,|\, F\sim G \in C\rangle$.

Example 9.6. Let $F = \min(0+x, 1+y, 2+z)$. $F_{\hat{x}} = \min(1+y, 2+z)$. So $F \sim F_{\hat{x}}$ is in $\mathcal{B}(F)$. For $w = (2,1,3)$, we have $\operatorname{in}_w F = \min(x,y)$ and $\operatorname{in}_w F_{\hat{x}} = y$. So $\operatorname{in}_w(F\sim F_{\hat{x}}) = \min(x,y) \sim y$.

Proposition 9.7 (Tropicalization and initial forms commute)**.**
(a) For $f \in k[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ and $w\in\mathbb{R}^n$, $\operatorname{in}_w(\operatorname{trop} f) = \operatorname{trop}(\operatorname{in}_w f)$.
(b) For $I \subset k[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ and $w\in\mathbb{R}^n$, $\operatorname{in}_w(\mathcal{B}(\operatorname{trop} I)) = \mathcal{B}(\operatorname{trop}(\operatorname{in}_w I))$.

We first observe the effect of a change of coordinates on tropical varieties and congruences. A monomial change of coordinates on $(k^\times)^n$, $\underline{x}^{\underline{u}} \mapsto \underline{x}^{A\underline{u}}$ for some $A \in GL_n(\mathbb{Z})$ corresponds to a map $F(\underline{x}) \mapsto F(A^T\underline{x}) =: A\cdot F$, where F is a tropical polynomial (cf. Lemma 3.2.7 in [MS]). For a congruence C on $\mathbb{T}[x_1^{\pm 1},\dots,x_1^{\pm n}]$ we define $A\cdot C = \{A\cdot F \sim A\cdot G \mid (F\sim G)\in C\}$. One can show that this action commutes with tropicalization: $\operatorname{trop}(V(A\cdot I)) = A\cdot \operatorname{trop}(V(I))$.

Proof of Theorem 9.4. Let $Y \subset (k^\times)^n$ with defining ideal I and let

$$\dim Y = \dim \operatorname{trop}(Y) = d.$$

We pick w in the relative interior of σ, where σ is a maximal cell of $\operatorname{trop}(V(I))$. Let $L = \operatorname{span}(w - w' \mid w' \in \sigma)$. After a change of coordinates we can assume that $L = \operatorname{span}(e_1,\dots,e_d)$.

By Lemma 3.3.6 in [MS] it follows that we have that

$$L = \operatorname{trop} V(\operatorname{in}_w I) = V(\operatorname{trop}(\operatorname{in}_w I)).$$

Indeed, if $\Sigma := \operatorname{trop}(V(I)) = \{w \mid \operatorname{in}_w I \neq (1)\}$ and σ a maximal cell and w in the relative interior of σ we have $L = \operatorname{star}_\Sigma(w) = \{v | \operatorname{in}_v(\operatorname{in}_w(I)) \neq (1)\}$.

We know from Proposition 9.7 that L can be recovered from $\mathcal{B}(\operatorname{trop}(\operatorname{in}_w I)) = \operatorname{in}_w(\mathcal{B}(\operatorname{trop} I))$. So L can be recovered from $\mathcal{B}(\operatorname{trop} I)$.

The initial ideal $\operatorname{in}_w I$ is homogeneous with respect to the grading $\deg(x_i) = e_i$ for $1 \le i \le d$ and $\deg(x_i) = 0$ otherwise. As remarked earlier, there is a generating set in $\bar{k}[x_{d+1}^{\pm 1},\dots,x_n^{\pm 1}]$ for $\operatorname{in}_w I \subset \bar{k}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$.

Consider $\mathrm{in}_w I \cap \bar{k}[x_1, \ldots, x_n]$ and denote by $\mathrm{in}_w I^h$ the homogenization of $\mathrm{in}_w I$ in $\bar{k}[x_0, \ldots, x_n]$. Recall that $\mathrm{mult}(\sigma) = \dim_{\bar{k}} \dfrac{\bar{k}[x_0, x_{d+1}, \ldots, x_n]}{\mathrm{in}_w I^h \cap \bar{k}[x_0, x_{d+1}, \ldots, x_n]}$. However, $\dfrac{\bar{k}[x_0, x_{d+1}, \ldots, x_n]}{\mathrm{in}_w I^h \cap \bar{k}[x_0, x_{d+1}, \ldots, x_n]}$ is zero-dimensional so the Hilbert polynomial of $\dfrac{\bar{k}[x_0, x_{d+1}, \ldots, x_n]}{\mathrm{in}_w I^h}$ must be a constant polynomial.

Recall that the Hilbert polynomial of a homogeneous ideal J can be recovered from $\mathcal{B}(\operatorname{trop} J)$. To show that $\mathrm{mult}(w)$ can be recovered from $\mathcal{B}(\operatorname{trop} I)$ it is enough to show that $\mathcal{B}(\operatorname{trop}(\mathrm{in}_w I^h))$ can be recovered from $\mathcal{B}(\operatorname{trop} I)$.

From Proposition 9.2 and Proposition 9.7 we know that

$$\mathcal{B}(\operatorname{trop}(\mathrm{in}_w I^h)) = \mathcal{B}(\operatorname{trop}(\mathrm{in}_w I))^h \quad \text{and} \quad \mathcal{B}(\operatorname{trop}(\mathrm{in}_w I))^h = \mathrm{in}_w(\mathcal{B}(\operatorname{trop} I))^h.$$

These imply that $\mathcal{B}(\operatorname{trop}(\mathrm{in}_w I^h)) = \mathrm{in}_w(\mathcal{B}(\operatorname{trop} I))^h$ which can be recovered from $\mathcal{B}(\operatorname{trop} I)$. □

References

[MS] D. Maclagan, B. Sturmfels *Introduction to tropical geometry*, Volume 161 of Graduate Studies in Mathematics AMS