TROPICAL BRILL-NOETHER THEORY

1. Divisors on Graphs

Let G be a graph. For example, consider the wedge of two triangles.

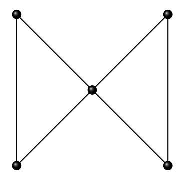

Figure 1. The wedge of two triangles.

Definition 1.1. *A* divisor D *(or* chip configuration *or* sandpile*) on a graph G is a formal $\mathbb{Z}$-linear combination of vertices of G,*

$$D = \sum_{v \in V(G)} D(v) \cdot v$$

with $D(v) \in \mathbb{Z}$.

For example, the following is a divisor on the wedge of triangles.

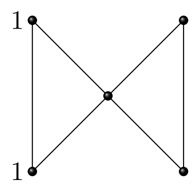

Figure 2. A divisor.

Divisors on graphs have been studied in combinatorics, computer science, and dynamics long before algebraic geometers got interested in them. In these disciplines it is more common to refer to divisors on graphs as *chip configurations* or *abelian sandpiles*. The term "chip configuration" comes from thinking of a divisor as a stack of poker chips on each vertex of the graph. Here we use the term *divisor* to emphasize the analogy with divisors on algebraic curves. Note that the divisors on a graph G form an abelian group, which we denote $\mathrm{Div}(G) = \mathbb{Z}^{V(G)}$.

Date: January 25, 2016. *Speaker*: David Jensen *Scribe*: Netanel Friedenberg.

We are interested in equivalence classes of divisors on graphs, where the equivalence is given by so-called chip-firing moves. Starting with a divisor, we may "fire" a vertex, which results in that vertex giving a chip to each of its neighbors. More concretely, we have the following definition.

Definition 1.2. *The* chip-firing move *at a vertex v takes a divisor D to D' where*

$$D'(w) = \begin{cases} D(v) - \mathrm{val}(v) & \text{if } w = v \\ D(v) + \#\text{ of edges between } w \text{ and } v & \text{if } w \neq v \end{cases}$$

In our example, if we fire the top left vertex, we get

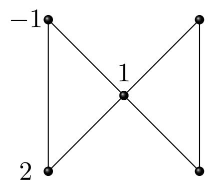

FIGURE 3. The result of firing a vertex.

Definition 1.3. *Two divisors D, D' are* linearly equivalent*, and we write $D \sim D'$, if D' can be obtained from D by a sequence of chip-firing moves.*

Lemma 1.4. *Linear equivalence of divisors is an equivalence relation. Moreover, if $D_1 \sim D_2$ and $E_1 \sim E_2$ then $D_1 + E_1 \sim D_2 + E_2$.*

Proof. To see that linear equivalence is reflexive, note that any divisor is equivalent to itself by the empty sequence of chip-firing moves.

To see that linear equivalence is symmetric, it suffices to show that a chip-firing move can be inverted by a sequence of chip-firing moves. To see this, note that firing every vertex other than a given vertex v is the inverse of firing v.

For transitivity, suppose that $D_1 \sim D_2$ and $D_2 \sim D_3$. Then, by concatenating the sequence of chip-firing moves that takes D_1 to D_2 with that which takes D_2 to D_3, we obtain a sequence of chip-firing moves that takes D_1 to D_3. Hence, $D_1 \sim D_3$.

Finally, by concatenating the sequence of chip-firing moves that takes D_1 to D_2 with that which takes E_1 to E_2, we obtain a sequence of chip-firing moves that takes $D_1 + E_1$ to $D_2 + E_2$. □

Definition 1.5. *The* degree *of a divisor $D = \sum_{v \in V(G)} D(v)v$ is the integer*

$$\deg(D) = \sum_{v \in V(G)} D(v).$$

Note that the degree is invariant under chip-firing.

Definition 1.6. *The* Picard group *of a graph G is the group of linear equivalence classes of divisors on G. That is,*

$$\mathrm{Pic}(G) = \mathrm{Div}(G)/\{\textit{divisors equivalent to } 0\}.$$

A divisor that is equivalent to 0 is called a principal divisor*.*

The degree is a group homomorphism $\mathrm{Pic}(G) \xrightarrow{\deg} \mathbb{Z}$. It is easy to see that this map is surjective, and the kernel is the group of divisors of degree 0. In other words, we have the short exact sequence

$$0 \to \mathrm{Pic}^0(G) \to \mathrm{Pic}(G) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Definition 1.7. *The* Jacobian $\mathrm{Jac}(G)$ *of a graph G is the group of linear equivalence classes of divisors of degree 0 on G. (The Jacobian is also known as the* sandpile group*, or* critical group*, and probably many other things besides.)*

Note that, since $\mathbb{Z}$ is free, the exact sequence above splits, so

$$\mathrm{Pic}(G) \cong \mathbb{Z} \oplus \mathrm{Jac}(G).$$

It follows that the degree d part of the Picard group, $\mathrm{Pic}^d(G)$, is a $\mathrm{Jac}(G)$-torsor. That is, the action of $\mathrm{Jac}(G)$ on $\mathrm{Pic}^d(G)$ by addition is free and transitive. We will see shortly how to compute Jacobians of graphs in general, but first let's look at some examples.

Example 1.8. On a tree, any two vertices are equivalent to each other. To see this, let T be a tree and $v, w \in V(T)$. Since T is a tree, there is a unique path from any vertex of T to w. Let $w = v_1, v_2, \ldots v_n = v$ be the unique path from w to v. We will prove by induction on n that $v_n \sim w$. To see this, fire every vertex v' such that the unique path from v' to w passes through v. Along any edge connecting two vertices in this firing set, there is a net change of 0 chips. It follows that firing this set takes v to v_{n-1}. By induction we have that $v_{n-1} \sim w$, hence $v \sim w$.

From this we see that $\mathrm{Jac}(T) = 0$, and so $\mathrm{Pic}(T) \cong \mathbb{Z}$.

Example 1.9. Let G be a cycle with n vertices. We will show that, unlike the case of a tree, the Jacobian of G is nontrivial. To see this, label the vertices (in order) with elements of $\mathbb{Z}/n\mathbb{Z}$, as pictured.

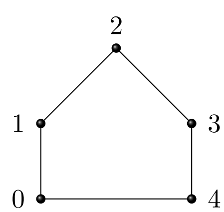

FIGURE 4. Labeling the vertices of a cycle.

Now consider the map $\mathrm{Div}(G) \to \mathbb{Z}/n\mathbb{Z}$ given by

$$\sum_{i=0}^{n-1} D(v_i)v_i \mapsto \sum_{i=0}^{n-1} D(v_i)i (\mod n).$$

Note that the expression on the right is unaffected by firing the vertex v_i, hence this map is invariant under linear equivalence. It follows that this descends to a map $\mathrm{Jac}(G) \to \mathbb{Z}/n\mathbb{Z}$.

To see that this map is surjective, note that we can put a -1 at the vertex labeled 0 and a 1 at the vertex labeled i for any i. It follows that $\mathrm{Jac}(G)$ is nontrivial. We will see later that this map is in fact injective as well. That is, $\mathrm{Jac}(G) \cong \mathbb{Z}/n\mathbb{Z}$.

Example 1.10. Let $G = G_1 \vee_v G_2$ be the wedge of two graphs. We have a map $\mathrm{Div}^0(G_1) \oplus \mathrm{Div}^0(G_2) \xrightarrow{\cong} \mathrm{Div}^0(G)$ by adding the divisors, and it is straightforward to see that this map is an isomorphism. To see that this isomorphism respects linear equivalence, note that the divisor obtained by firing any vertex other than v in the graph G_1 maps to the divisor obtained by firing that same vertex in G, whereas the divisor obtained by firing v in G_1 maps to the divisor obtained by firing all vertices in G_2.

It follows that $\mathrm{Jac}(G_1) \oplus \mathrm{Jac}(G_2) \xrightarrow{\cong} \mathrm{Jac}(G)$. Recalling our example of the wedge of two triangles, we see that $\mathrm{Jac}(G) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

We now turn to the problem of computing Jacobians of graphs more generally.

Definition 1.11. *The* graph Laplacian *of a graph G is the square matrix with rows and columns indexed by the vertices of G, and whose (i, j)th entry is*

$$\Delta_{i,j} = \begin{cases} -\mathrm{val}(v_i) & \text{if } i = j \\ \#\text{ of edges between } v_i \text{ and } v_j & \text{if } i \neq j \end{cases}$$

That is, Δ is the difference of the adjacency matrix and the valency matrix.

Example 1.12. Consider the following graph[1]

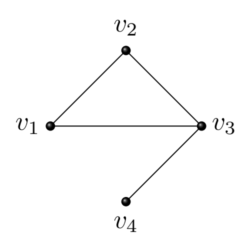

FIGURE 5. A simple graph.

The graph Laplacian of this graph is then

$$\Delta = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

If we consider a divisor as a vector in $\mathbb{Z}^{V(G)}$, then Δe_i is the divisor obtained from the identically zero divisor by firing the vertex v_i. More generally, for any vector $f \in \mathbb{Z}^{V(G)}$, Δf is the divisor obtained from 0 by firing each vertex v_i $f(i)$

[1] In the film "Good Will Hunting", the first of the two problems to appear on the blackboard is a four-parter, the first part of which is to compute the Laplacian of this graph.

times. Hence $\text{Im}(\Delta)$ is exactly the set of divisors equivalent to 0. It follows that $\mathbb{Z}^{V(G)}/\text{Im}(\Delta) = \text{Pic}(G)$.

Note that $\det(\Delta) = 0$, because the sum of the columns of Δ is zero. From this we see that $\text{Pic}(G)$ is infinite. Of course, we could have deduced this previously from the fact that the degree homomorphism maps $\text{Pic}(G)$ surjectively onto the integers. The order of the Jacobian is the absolute value of the determinant of the *reduced Laplacian*, which is the matrix $\tilde{\Delta}$ obtained by removing any row from Δ and the corresponding column.

So in our example, the reduced Laplacian obtained by removing the third row and column is

$$\tilde{\Delta} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The determinant of $\tilde{\Delta}$ is -3, so $\text{Jac}(G) \cong \mathbb{Z}/3\mathbb{Z}$. Note that we can see this from the examples above, because this graph is the wedge of a triangle and a tree.

In other words, $\text{Jac}(G) \cong \mathbb{Z}^{V(G)-1}/\text{Im}(\tilde{\Delta})$. In this way, we see that Jacobians of graphs are easily computable. Indeed, if one puts the reduced Laplacian in Smith normal form, one obtains a decomposition of $\text{Jac}(G)$ as a direct sum of cyclic groups.

Recall that a *spanning tree* in a graph G is a subgraph that contains every vertex and is a tree. Perhaps the most well-known result concerning reduced graph Laplacians is Kirchoff's Matrix Tree Theorem, which we will not prove here.

Theorem 1.13. *(Matrix Tree Theorem) The absolute value of the determinant of the reduced graph Laplacian of a graph G is equal to the number of spanning trees in G.*

Corollary 1.14. *For any graph G, the order of* $\text{Jac}(G)$ *is equal to the number of spanning trees in G. In particular,* $\text{Jac}(G)$ *is a finite abelian group.*

Returning to Example 1.9, note that a cycle G with n edges has precisely n spanning trees, each obtained by removing one edge. It follows that the order of $\text{Jac}(G)$ is equal to n. Since we have seen that $\text{Jac}(G)$ maps surjectively onto $\mathbb{Z}/n\mathbb{Z}$, we see that $\text{Jac}(G) \cong \mathbb{Z}/n\mathbb{Z}$.

We now turn to a property that is central to our purposes. Everything that we have discussed so far is standard material in any discipline that studies graph Jacobians, including combinatorics and dynamical systems. The following definition, however, is motivated purely by algebraic geometry.

Definition 1.15. *A divisor $D = \sum_{v \in V(G)} D(v)v$ on a graph G is effective if $D(v) \geq 0$ for all $v \in V(G)$.*

Note that effectiveness is *not* invariant under chip-firing. In our first example, the divisor pictured in Figure 2 is effective, but the linearly equivalent divisor pictured in Figure 3 is not. Next time, we will see an algorithm for computing whether a given divisor is linearly equivalent to an effective divisor.

Definition 1.16. *A divisor D on a graph G is said to have* rank *at least r if, for every effective divisor E of degree r, $D - E$ is equivalent to an effective divisor.*

The rank of a divisor D is the largest non-negative integer such that D has rank at least r, and D has rank -1 if no such r exists.

By definition, a divisor has nonnegative rank if and only if it is equivalent to an effective divisor.

Example 1.17. We show that, if D is a divisor of degree $r \geq 0$ on a tree, then $\mathrm{rk}(D) = \deg(D) = r$. To see this, note that if E is an effective divisor of degree r, then since D and E are equivalent by Example 1.8, we have $D - E \sim 0 \geq 0$. It follows that $\mathrm{rk}(D) \geq \deg(D)$. Since there are no effective divisors of negative degree, we see that $\mathrm{rk}(D)$ is exactly $\deg(D)$.

References

[BN07] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.