TROPICAL BRILL-NOETHER THEORY

12. Tropical proof of the Brill-Noether theorem

Based on "A Tropical Proof of the Brill-Noether Theorem" by Cools, Draisma, Payne, and Robeva.

In this lecture, we will study the varieties $W^r_d(X)$ parameterizing divisors of degree d and rank at least r on a general curve X. Our aim is to prove the Brill-Noether theorem.

Theorem 12.1 (Brill-Noether theorem)**.** *Let X be a general curve of genus g.*

(1) If $\rho(g,r,d) := g-(r+1)(g-d+r) < 0$, then $W^r_d(X)$ is empty.
(2) If $\rho(g,r,d) \geq 0$, then $\dim(W^r_d(X)) = \min\{g, \rho(g,r,d)\}$.

The Brill-Noether theorem differs from more classical results in the theory of algebraic curves, such as Riemann-Roch or Clifford's Theorem, in that it applies only to a general curve, rather than all curves. The Brill-Noether theorem was first proven in 1980 in a seminal paper of Griffiths and Harris, and has subsequently been re-proven by different authors using a variety of different techniques. In this lecture, we will provide a proof of the Brill-Noether theorem using the theory of specialization of divisors from curves to graphs. This will mark the first point, in these lectures, that we use the techniques of divisors on graphs to prove a theorem about the geometry of algebraic curves.

The lower bound

$$\dim W^r_d(X) \geq \rho(g,r,d)$$

holds for all curves of genus g, and was established independently by Kempf and Kleiman-Laksov. To do this, they exhibit $W^r_d(X)$ as a determinantal variety, and then prove that a certain vector bundle satisfies a type of positivity. Our goal in this lecture will be to establish the upper bound

$$\dim W^r_d(X) \leq \rho(g,r,d)$$

for general curves of genus g. By semicontinuity, it suffices to construct a single curve of genus g for which this inequality holds.

We first recall some facts from Lecture 5, where we studied the special divisors on a generic chain of loops G. Recall that the upper and lower edge lengths of the ith loop are denoted by ℓ_i and m_i, respectively, and the chain of loops is called generic if, for all i, the number $\frac{\ell_i}{m_i}$ cannot be expressed as a ratio of two integers whose sum has absolute value smaller than $2g-2$. In Lecture 5, it is shown that, to any divisor $D \in \mathrm{Pic}(G)$, one can associate a lingering lattice path $P_0, \ldots, P_g \in \mathbb{Z}^r$. The divisors of rank at least r can then be characterized using these lattice paths.

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Theorem 12.2 (Theorem 5.12)**.** *A divisor D on G has rank at least r if and only if its associated lingering lattice path lies entirely in*

$$\mathcal{C} = \{y \in \mathbb{Z}^r | y(0) > y(1) > \cdots > y(r-1) > 0\}.$$

We will first show an analogue of the Brill-Noether theorem for the generic chain of loops G.

Theorem 12.3. *Let G be a generic chain of loops.*

1. *If $\rho(g, r, d) < 0$, then G has no divisors of degree d and rank at least r.*
2. *If $\rho(g, r, d) \geq 0$, then G has no divisors D of degree d and rank at least r such that $D \geq (r + \rho + 1)v_0$.*

Proof. Let $D \in W^r_d(G)$, and assume without loss of generality that D is v_0-reduced. Let $d_0 = \deg_{v_0} D$. Since D is v_0-reduced, it cannot have more then one chip on each loop, so exactly $d - d_0$ of the loops contain a chip of D. It follows that there are exactly $g - d + d_0$ empty loops. From this it follows that exactly $g - d + d_0$ steps of the corresponding lingering lattice path are in the direction $(-1, \ldots, -1)$.

By definition of $\mathcal{C}$, we see that the number of steps in each coordinate direction e_i must therefore be greater than or equal to $g - d + r$. Since the total number of steps is g, we must have

$$g \geq (g - d + d_0) + r(g - d + r).$$

It follows that

$$\rho(g, r, d) = g - (r + 1)(g - d + r) \geq d_0 - r,$$

proving part (2). Since D has rank at least r, it follows that $d_0 \geq r$, proving part (1). □

Combining this result with Baker's specialization lemma will yield the Brill-Noether theorem.

Proof of the Brill-Noether theorem. Let X be a curve of genus g over a discretely valued field K, and let $\mathfrak{X}$ be a strongly semistable model for X such that the dual graph of the cental fiber is a generic chain of loops G. We note that every effective divisor of degree $r + \dim W^r_d(X)$ is contained in an effective divisor whose class is in $W^r_d(X)$. Let $x \in X$ be a point that specializes to $v_0 \in V(G)$, and let $D \in W^r_d(X)$ be a divisor such that $D \geq (r + \dim W^r_d(X))x$. Note that $\deg_{v_0} \rho(D) \geq r + \dim W^r_d(X)$.

The divisor D is defined over some finite extension K' of K. If the ramification index of the extension K'/K is e, then there exists a strongly semistable model $\mathfrak{X}'$ for X over the valuation ring of K' such that the dual graph of the central fiber is the refinement $\frac{1}{e}G$, obtained by subdividing each edge of G into e edges. Note that $\frac{1}{e}G$ is also a generic chain of loops, so we may assume without loss of generality that D is defined over K.

By Baker's specialization lemma (Lemma 7.12), $r(\rho(D)) \geq r$. By Theorem 12.3, if $\rho(g, r, d) < 0$, then the graph G has no divisors of degree d and rank at least r. It follows that $\rho(g, r, d) \geq 0$, proving part (1) of the Brill-Noether theorem. Similarly, by Theorem 12.3, if $\rho(g, r, d) \geq 0$, then G has no divisors D of degree d and rank at least r such that $D \geq (r + \rho + 1)v_0$. It follows that $\dim W^r_d(X) \leq \rho(g, r, d)$, proving part (2) of the Brill-Noether theorem. □

References

[Bak08] M. Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008.

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