TROPICAL BRILL-NOETHER THEORY

13. Tropical independence and proof of the Gieseker-Petri theorem (for $\rho = 0$)

The goal of this lecture is a tropical proof the Gieseker-Petri theorem, due to Jensen and Payne, in the $\rho = 0$ case.

Let X be a general curve of genus g over K a complete non-archimedean field. The scheme $\mathcal{G}^r_d(X)$ parameterizes linear series of degree d and rank r on X. In the last lecture we saw the Brill-Noether theorem, which says that $\mathcal{G}^r_d(X)$ has pure dimension $\rho = g - (r+1)(g-d+r)$ if $\rho \geq 0$, and is empty if $\rho < 0$.

Theorem 13.1 (Gieseker-Petri)**.** *The scheme $\mathcal{G}^r_d(X)$ is smooth.*

It is an established fact about algebraic curves that for a linear series $W \subset \mathcal{L}(D_X)$,

$$\dim T_W(\mathcal{G}^r_d(X)) = \rho + \dim \ker \mu_W,$$

where $\mu_W : W \otimes \mathcal{L}(K_X - D_X) \to \mathcal{L}(K_X)$ is the multiplication map. So, to prove the theorem, we must show that μ_W is injective for all such D_X and W. It is enough to show the following result:

Theorem 13.2. *If $\Sigma(X^{\mathrm{an}})$ is a generic chain of loops with bridges, then the map $\mu : \mathcal{L}(D_X) \otimes \mathcal{L}(K_X - D_X) \to \mathcal{L}(K_X)$ is injective for all divisors D_X.*

The proof will combine the tropicalization maps and specialization lemma which we have studied together with combinatorics of divisors on chains of loops. Let Γ a metric graph. Recall that $\mathrm{PL}(\Gamma)$ is the set of piecewise linear function on Γ with integer slopes, analogous to rational functions. Given a divisor D on Γ, $|D|$ is the complete linear series of D, that is, the set of effective divisors equivalent to D. Equivalently,

$$|D| = \{D + \mathrm{div}(\phi) \geq 0 : \phi \in \mathrm{PL}(\Gamma)\}.$$

This object is analogous to the complete linear series $|D_X|$ on X.

For a divisor D, define $R(D) = \{\phi \in \mathrm{PL} : D + \mathrm{div}(\phi) \geq 0\}$, analogous to the space of rational functions on X with zeros and poles of at most the orders prescribed by a divisor D_X at each point. Observe that we have a map $R(D) \to |D|$ given by $\phi \mapsto D + \mathrm{div}(\phi)$. The set $R(D)$ has the structure of a tropical module, analogous to the vector space structure on $\mathcal{L}(D_X)$. This means that for $\phi_1, \ldots, \phi_n \in R(D)$ and $b_1, \ldots, b_n \in \mathbb{R}$, we have $\min_i(\phi_i + b_i) \in R(D)$. In other words, $R(D)$ is closed under "tropical linear combinations".

Recall the tropicalization maps on divisors and rational functions

$$\mathrm{Trop} : \mathrm{Div}(X) \to \mathrm{Div}(\Gamma)$$

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and

$$\mathrm{trop} : K(X)^* \to \mathrm{PL}(\Gamma),$$

the latter given by $f \mapsto (y \mapsto \mathrm{val}_y(f))$. The Poincare-Lelong formula shows that $\mathrm{Trop}(\mathrm{div}(f)) = \mathrm{div}(\mathrm{trop}(f))$. We have also seen the Specialization Lemma, which says that $r(\mathrm{Trop}(D_X)) \geq r(D_X)$, where $r(\mathrm{Trop}(D_X))$ is the combinatorial rank of a divisor on a metric graph and $r(D_X)$ is algebraic rank of a divisor on a curve, i.e., $h^0(\mathcal{L}(D_X)) - 1$. This inequality is not an equality, in general. For example, effective divisors on X stay effective on Γ, but non-effective divisors might become effective by cancellation.

We now show how to use the tropical module structure discussed above to prove statements about linear independence of divisors.

Definition 13.3. *The set of functions* $\{\phi_1, \ldots, \phi_n\} \subseteq \mathrm{PL}(\Gamma)$ *is* tropically dependent *if there exist* $b_1, \ldots, b_n \in \mathbb{R}$ *such that* $\min_i(\phi_i + b_i)$ *occurs at least twice at every point of* Γ.

Example 13.4. On the segment $[-1, 1]$, the two functions $x \mapsto 2x$ and $x \mapsto 2x+1$ are tropically dependent. Less trivially, the three functions $x \mapsto |x|$, $x \mapsto \max\{-x, 2x\} + 37$, and $x \mapsto \max\{-2x, x\} - 2016$ are tropically dependent.

Remark 13.5. This notion of linear dependence does not have matroidal properties. Indeed, one can construct a set of piecewise linear functions with maximal tropically independent subsets of different cardinalities.

Observe that if $f_1, \ldots, f_r \in K(X)$ are linearly dependent and $\phi_i = \mathrm{trop}(f_i)$, then $\phi_1, \ldots, \phi_r$ are tropically dependent. This is clear: write $\sum_i c_i f_i = 0$, where not all of the c_i are 0, and take valuations, using the non-archimedean property that $v(a + b) = \min(v(a), v(b))$ if $v(a) \neq v(b)$.

Consider the map $\mu : \mathcal{L}(D_X) \otimes \mathcal{L}(E_X) \to \mathcal{L}(D_X + E_X)$. Choose bases $\{f_i\}$ and $\{g_j\}$ for $\mathcal{L}(D_X)$ and $\mathcal{L}(E_X)$, respectively. Let $\phi_i = \mathrm{trop}(f_i)$ and $\psi_j = \mathrm{trop}(g_j)$. The following lemma follows immediately from the preceding observation:

Lemma 13.6. *If* $\{\mathrm{trop}(f_i g_j)\} = \{\phi_i + \psi_j\}$ *is tropically independent, then* μ *is injective.*

We will apply this lemma to piecewise linear functions on the chain of g loops with generic edge lengths in the special case when $\rho = 0$. This is the only case we will present, and does not require bridges between the loops; the cases where $\rho > 0$ are more complicated, requiring additional techniques from Berkovich theory and reductions of rational functions, and the proof when $\rho > 0$ does require bridges of nonzero length between the loops.

Fix a divisor D_X. We shall apply the lemma to $\mathcal{L}(D_X)$ and $\mathcal{L}(K_X - D_X)$ to prove the theorem.

Let Γ be a chain of g loops with generic edge lengths. The ith loop has a left endpoint v_i and a right endpoint w_i, and there are bridges br_i connecting w_i to v_{i+1}. We denote the ith loop by γ_i, excluding the right endpoint w_i. The edge lengths ℓ_i and m_i on the loop γ_i are *generic* if there do not exist integers a and b with $a + b \leq 2g - 2$ and $\frac{\ell_i}{m_i} = \frac{a}{b}$.

The outline will be:

(1) Show that any divisor in $|K|$ contains no point in one of $\gamma_1, \ldots, \gamma_g$.
(2) Show that, for $D \in \operatorname{Div}\Gamma$, $\psi_0, \ldots, \psi_r \in R(D)$, $\theta = \min_i \psi_i$, and $\Gamma' \subseteq \Gamma$ connected, if $D + \operatorname{div}(\psi_i)$ contains a chip in Γ' for each i, then so does $D + \operatorname{div}\theta$.
(3) Construct representatives of $|D|$ and $|K - D|$, where $D = \operatorname{trop} D_X$ and use (1) to show that the corresponding functions in $R(K)$ are tropically independent.

Note that, by the lemma above, (3) implies that the map μ is injective, so once this is established, we will be done.

We first show that (3) follows from (1) and (2). To see this, we shall suppose to the contrary that that these functions are tropically dependent and use (2) to construct a divisor in the class of K with a point on each γ_i, contradicting (1).

Recall the bijection between v_1-reduced divisors of degree d and rank r on Γ and standard tableaux of shape $(r+1) \times (g-d+r)$ via the bijection of each with lingering lattice paths. It turns out that the involution $D \leftrightarrow K - D$ on divisors corresponds to transposition on tableaux, $P \leftrightarrow P^T$.

Lemma 13.7. *Suppose D is a divisor of degree d and rank r on Γ and let P be the tableau corresponding to D.*

(a) There are unique representatives $D_0, \ldots, D_r \in |D|$ such that for each j, $D_j - j[v_1] - (r-j)[w_g]$ is effective, and D_j has no chips on γ_i if and only if i is in the j-th column of P.
(b) There are unique representatives $E_0, \ldots, E_{g-d+r-1} \in |K - D|$ such that $E_j - j[v_1] - (g-d+r-1-j)[w_j]$ and E_j has no chips on γ_i if and only if i is in the j-th row of P.
(c) The divisor $D_j + E_k \in |K|$ has no chips on γ_i if and only if i is the (j,k) entry of P.

Proof. The proof of (a) is a chip-firing argument: start with j chips at v_0, fire them left to right, and consider at which loops you pick up a chip. Part (b) follows from (a) by applying the involution. Part (c) follows from (a) and (b). □

Proof of (3), assuming (1) and (2). By construction, there exist bases $\{f_j\}$ and $\{g_k\}$ for $\mathcal{L}(D_X)$ and $\mathcal{L}(K_X - D_X)$, respectively, such that $\operatorname{Trop}(D_X + \operatorname{div}(f_j)) = D_j$ and $\operatorname{Trop}(K_X - D_X + \operatorname{div}(g_k)) = E_k$. Let $\phi_j = \operatorname{trop}(f_j) \in R(D)$ and $\psi_k = \operatorname{trop}(g_k) \in R(K - D)$, so $D_j = D + \operatorname{div}(\phi_j)$ and $E_k = E + \operatorname{div}(\psi_k)$.

We claim that the set $\{\phi_j + \psi_k\}$ is tropically independent. Suppose it is not. By modifying these functions by constants, we may assume $\theta = \min_{j,k}(\phi_j + \psi_k)$ is achieved at least twice at every point. We show $K + \operatorname{div}\theta$ has a chip on each loop γ_i.

Fix i and suppose i is the (j_0, k_0) entry of the tableau corresponding to D. Because the minimum is achieved twice at every point, $\theta = \min_{(j,k)\neq(j_0,k_0)}(\phi_j + \psi_k)$. Each $K + \operatorname{div}(\phi_j + \psi_k) = D_j + E_k$, where $(j,k) \neq (j_0, k_0)$, has a chip in γ_i by part (c) of Lemma 13.7. By (2), so does $K + \operatorname{div}\theta$.

Therefore, $K + \operatorname{div}\theta$ has a chip on each loop of Γ. But $K + \operatorname{div}\theta \in |K|$, which contradicts (1). □

Proof of (1). Fix $D \in |K|$. Write $D = K + \operatorname{div}\psi$ for $\psi \in R(K)$. Let Γ_ψ be the closed set where ψ achieves its minimum. The set Γ_ψ is a union of edges of Γ with no leaves. Indeed, if Γ_ψ contains an edge e with leaf u, let v be the other end of e. Then

$K_\Gamma(v) = \deg v - 2$, while $\mathrm{ord}_v(\psi) \leq -(\deg v - 1)$, since ψ is constant on e and has slope at least 1 on all other edges incident to v. So $D(v) = K_\Gamma(v) + \mathrm{ord}_v(\psi) \leq -1$, a contradiction.

So Γ_ψ is a union of loops, bridges, and halves of loops. Let i be minimal such that $\gamma_i \subset \Gamma_\psi$. Then Γ_ψ is disjoint from $\gamma_1, \ldots, \gamma_{i-1}$, and br_{i-1}. Therefore, D has no chips on γ_i, since

$$ord_{v_i}(\psi) \geq \begin{cases} 1 & i \geq 1, \\ 0 & i = 1) \end{cases} = K_\Gamma(v_i).$$

□

Lemma 13.8. *Suppose $D \in \mathrm{Div}(\Gamma)$, $\psi_0, \ldots, \psi_r \in R(D)$, $\theta = \min_i \psi_i$. Let*

$$\Gamma_i = \{v \in \Gamma : \theta(v) = \psi_i(v)\}.$$

Then

$$\mathrm{supp}(D + \mathrm{div}(\theta)) \cap \Gamma_i = (\mathrm{supp}(D + \mathrm{div}(\psi_i)) \cap \Gamma_i) \cup \partial\Gamma_i.$$

Proof of (2). Fix $D \in \mathrm{Div}\,\Gamma$ and $\psi_0, \ldots, \psi_r \in R(D)$, and let θ and Γ_i be as defined.

Choose i such that $\theta(v) = \psi_i(v)$ for some $v \in \Gamma'$. Let $\Gamma'_i = \{v \in \Gamma' : \theta(v) = \psi_i(v)\}$. We have $\Gamma'_i \neq \emptyset$ by our choice of i. If $\Gamma'_i = \Gamma'$, then $D + \mathrm{div}(\psi_i) = D + \mathrm{div}(\theta)$ on Γ', so $D + \mathrm{div}(\theta)$ has a chip on Γ' because $D + \mathrm{div}(\psi_i)$ does. Otherwise, by Lemma 13.8, $D + \mathrm{div}(\theta)$ has a chip at some point in $\Gamma' \cap \partial\Gamma'_i$. □

References

[JP14] D. Jensen and S. Payne. Tropical independence I: Shapes of divisors and a proof of the Gieseker-Petri theorem. *Algebra Number Theory*, 8(9):2043–2066, 2014.