TROPICAL BRILL-NOETHER THEORY

14. Uniformization of curves and Jacobians

We will proceed from the classical theory of complex analytic uniformization of elliptic curves to the p-adic analogue introduced by Tate [Tat95]. Then we will move on to discuss Mumford's theory of p-adic analytic uniformization of curves of higher genus [Mum72b, Mum72a]. Finally, we will end up with the classical and p-adic uniformization of abelian varieties and Jacobians of curves and some musings about more general higher dimensional varieties.

In addition to the original works of Mumford and Tate, additional references include [Bak08, §5.2], [Ber90, §4.4], [GvdP80, §I.2 and III.2], and [Lü16, §2.8].

14.1. **Uniformization and the Tate curve.** We begin with the classical uniformization theorem for compact Riemann surfaces over $\mathbb{C}$, presenting any compact Riemann surface as a quotient of its universal cover (which is either the Riemann sphere, complex plane, or complex upper half) by the action of a discrete subgroup of its group of transformations.

Theorem 14.1 (Uniformization Theorem)**.** *Let C be a compact Riemann surface. Then either:*

- *$C \cong \mathbb{P}^1$ is the Riemann sphere.*
- *C is an elliptic curve. In this case, $C(\mathbb{C}) \cong \mathbb{C}/\Lambda$ where $\Lambda \subset \mathbb{C}$ is a lattice, and we can take $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ with τ in the upper half-plane $\mathbb{H}$.*
- *C has genus ≥ 2. In this case, $C(\mathbb{C}) \cong \mathbb{H}/\Gamma$ where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a discrete subgroup, i.e., a* Fuchsian *group.*

What is the p-adic analogue (i.e., for smooth proper curves over $\mathbb{C}_p$) of the uniformization theorem? Here, $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$.

For elliptic curves, a naive imitation of the uniformization in the complex case fails over a p-adic field. Indeed, if we try $\mathbb{C}_p/\Lambda$ where $\Lambda \subset \mathbb{C}_p$ is an additive subgroup, the problem is that Λ cannot be discrete, because $p^n\lambda \to 0$ for any $\lambda \in \Lambda$. However, we can reinterpret the uniformization in the case of elliptic curves using the exponential map

$$\begin{tikzcd} \mathbb{C} \arrow[r, "e^{2\pi i \bullet}"] \arrow[d, "\Lambda"'] & \mathbb{C}^\times \arrow[d] \\ \mathbb{C}/\Lambda \arrow[r, "\cong"] & \mathbb{C}^\times/q^{\mathbb{Z}} \end{tikzcd}$$

where $q = e^{2\pi i\tau}$ satisfies $|q| < 1$ since $\tau \in \mathbb{H}$.

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While $\mathbb{C}^\times$ is not simply connected, hence is not the universal cover of an elliptic curve over $\mathbb{C}$, the p-adic analogue $\mathbb{C}_p^\times$ is simply connected. Tate realized that imitating this exponential uniformization for a p-adic elliptic curve works.

Theorem 14.2 (Tate)**.** *For any $q \in \mathbb{C}_p^\times$ with $|q| < 1$, there exists an elliptic curve E with multiplicative reduction defined over $\mathbb{C}_p$ and a "p-adic analytic" uniformization isomorphism*

$$\mathbb{C}_p^\times / q^{\mathbb{Z}} \xrightarrow{\sim} E(\mathbb{C}_p).$$

If $q \in \mathbb{Q}_p^\times$ then E is defined over $\mathbb{Q}_p$ with split multiplicative reduction and the uniformization isomorphism is "Galois equivariant", i.e., for any finite Galois extension $K/\mathbb{Q}_p$, the isomorphism

$$K^\times / q^{\mathbb{Z}} \xrightarrow{\sim} E(K)$$

is equivariant under the natural action of the Galois group of $K/\mathbb{Q}_p$ on both sides.

Conversely, any elliptic curve E defined over $\mathbb{Q}_p$ with split multiplicative reduction admits a uniformization $\mathbb{C}_p^\times / q^{\mathbb{Z}} \xrightarrow{\sim} E(\mathbb{C}_p)$ for some $q \in \mathbb{Q}_p$ with $|q| < 1$.

We recall that an elliptic curve E defined over a valued field K, with valuation ring R and residue field k, has *multiplicative reduction* if E admits a proper flat model $\mathcal{E}$ over R whose special fiber $\mathcal{E}_k$ has a single node whose complement (i.e., the smooth locus of the special fiber) is a 1-dimensional affine algebraic torus over k. Multiplicative reduction is called *split* if the smooth locus of the special fiber is isomorphic to $\mathbb{G}_m$.

The Tate uniformization works on the level of Berkovich analytic spaces and shows that the analytic space E^{an} of an elliptic curve E with split multiplicative reduction is locally bianalytic to $(\mathbb{A}^1)^{\mathrm{an}}$. On the contrary, if E has good reduction then E^{an} contains a type 2 point x such that $\mathcal{H}(x)$ is the function field of a genus 1 curve, and hence E^{an} is not locally bianalytic to $(\mathbb{A}^1)^{\mathrm{an}}$. In this case, E^{an} is contractible and, in fact, is homeomorphic to $(\mathbb{P}^1)^{\mathrm{an}}$,

Remark 14.3. , we now explain how isomorphism classes of 1-dimensional affine algebraic tori, other than $\mathbb{G}_m$, over a field k are in bijection with the isomorphism classes of separable quadratic extensions l/k, as follows. The *Weil restriction torus* $R_{l/k}\mathbb{G}_m$ is the 2-dimensional affine algebraic torus over k with functor of points

$$R_{l/k}\mathbb{G}_m(A) = \mathbb{G}_m(A \otimes_k l)$$

for a k-algebra A. There exists a norm homomorphism

$$\mathbb{R}_{l/k}\mathbb{G}_m \xrightarrow{N} \mathbb{G}_m$$

induced from the usual norm map $N : l \to k$, taking $\mathbb{G}_m(A \otimes_k l)$ to $\mathbb{G}_m(A)$. Its kernel, denoted $R^1_{l/k}\mathbb{G}_m$, is a 1-dimensional affine algebraic torus over k. Conversely, any 1-dimensional affine algebraic torus defined over k is either $\mathbb{G}_m$ or is of this form.

In equations, $\mathbb{G}_m \subset \mathbb{A}^2$ is defined by $xy = 1$. For simplicity, assume for the moment that k does not have characteristic 2, so that after a change of variables, $\mathbb{G}_m$ is defined by $x'^2 - y'^2 = 1$. If we write $l = k(\sqrt{d})$, then $R^1_{l/k}\mathbb{G}_m \subset \mathbb{A}^2$ is defined by the equation $x'^2 - dy'^2 = 1$.

There are two examples to keep in mind. The first is $k = \mathbb{R}$, in which case there are two isomorphism classes of 1-dimensional affine algebraic tori: $\mathbb{G}_m$ and

$R^1_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ = "algebraic S^1" $= \{x^2 + y^2 = 1\}$. The second example is $k = \mathbb{F}_q$, in which case there are again two isomorphisms classes of 1-dimensional affine algebraic tori: $\mathbb{G}_m$ and $R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m$. These are distinguished by their number of rational points: $\mathbb{G}_m(\mathbb{F}_q) = q - 1$ and $R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_m(\mathbb{F}_q) = q + 1$.

14.2. **Mumford uniformization.** Mumford found a good generalization of Tate's uniformization to curves of higher genus. In this case, it is not even clear how to naively imitate the hyperbolic uniformization from the complex case: what is the p-adic analogue of $\mathbb{H}$ or its isometry group $\mathrm{PSL}_2(\mathbb{R})$? Mumford's perspective (also due to Drinfeld and Manin) is that instead of trying to imitate a hyperbolic uniformization, a Schottky uniformization is more amenable to a p-adic analogue.

A *Schottky group* is a discrete subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{C})$, acting on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ via Möbius transformations, defined by the following construction. Choose g pairs $A_1, B_1, A_2, B_2, \ldots, A_g, B_g$ of open (geometric) disks in $\mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$. Let $\gamma_i \in \mathrm{PGL}_2(\mathbb{C})$ be the unique Möbius transformation mapping the disks A_i, B_i into each other, i.e., satisfying $\gamma_i(\mathbb{C}\backslash A_i) = \overline{B_i}$ and $\gamma_i(\mathbb{C}\backslash B_i) = \overline{A_i}$. Any such transformation is *loxodromic*, i.e., is conjugate to a unique matrix $\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $|q| < 1$, and has two fixed points. Finally, the associated Schottky group of rank g is $\Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle \subset \mathrm{PGL}_2(\mathbb{C})$. By a theorem of Poincaré, Γ is free and a discrete subgroup of $\mathrm{PGL}_2(\mathbb{C})$. Now we consider how Γ acts on the Riemann sphere. A limit point of Γ is a point $x \in \mathbb{P}^1(\mathbb{C})$ such that there exists $y \in \mathbb{P}^1(\mathbb{C})$ and a sequence of transformations $\alpha_j \in \Gamma$ so that $\alpha_j(y) \to x$. Let $\Sigma_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ be the set of limit points of Γ. In fact, $\mathbb{P}^1(\mathbb{C})\backslash\Sigma_\Gamma$ is the largest subset on which Γ acts properly discontinuously, and then $(\mathbb{P}^1(\mathbb{C})\backslash\Sigma_\Gamma)/\Gamma$ is a compact Riemann surface genus g. Conversely, given any compact Riemann surface C of genus $g \geq 1$, there exists a Schottky group $\Gamma \subset \mathrm{PGL}_2(\mathbb{C})$ of genus g such that $(\mathbb{P}^1(\mathbb{C})\backslash\Sigma_\Gamma)/\Gamma \cong C(\mathbb{C})$.

As a (slightly degenerate) example, consider a Schottky group Γ of genus $g = 1$, i.e., we start with a pair of circles A_1, B_1 mapped into each other by γ_1 and then $\Gamma = \langle \gamma_1 \rangle$ is a free group of rank 1. In this case, the limit set $\Sigma_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ consists of two points (the fixed points of the loxodromic element γ_1), which we can call 0 and ∞. Also recall the complex number q with $|q| < 1$ associated to the loxodromic element γ_1. Finally, under these identifications, we have that $(\mathbb{P}^1(\mathbb{C})\backslash\Sigma_\Gamma)/\Gamma \cong \mathbb{C}^\times/q^{\mathbb{Z}}$. Thus we see that the multiplicative uniformization of an elliptic curve considered above is an example of a Schottky uniformization.

As we remarked before, while $\mathbb{P}^1(\mathbb{C})\backslash\Sigma_\Gamma$ is not simply connected, hence is not the universal cover of a Riemann surface, the p-adic analytic analogue $\mathbb{P}^1(\mathbb{C}_p)\backslash\Sigma_\Gamma$ is simply connected. Motivated by the complex Schottky uniformization, Mumford defines a *p-adic Schottky group* to be a discrete, finitely generated, free subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_p)$ whose limit set $\Sigma_\Gamma \subset \mathbb{P}^1(\mathbb{C}_p)$ of the action on $\mathbb{P}^1(\mathbb{C}_p)$ is not everything. It turns out that $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_p)$ is a p-adic Schottky group if and only it it is discrete, finitely generated, and torsion free, and that the limit set is always defined over $\mathbb{Q}_p$, i.e., $\Sigma_\Gamma \subset \mathbb{P}^1(\mathbb{Q}_p)$.

The construction of p-adic Schottky groups is analogous to the complex case. Choose g pairs $A_1, B_1, A_2, B_2, \ldots, A_g, B_g$ of open disks in $\mathbb{P}^1(\mathbb{C}_p)$ whose centers are in $\mathbb{P}^1(\mathbb{Q}_p)$, whose radii satisfy $r(A_i)r(B_i) \in p^{\mathbb{Z}}$, and such that the associated closed disks

$A_1^+, B_1^+, \ldots, A_g^+, B_g^+$ are disjoint. Then there exist elements $\gamma_i \in \mathrm{PGL}_2(\mathbb{Q}_p)$ with $\gamma_i(\mathbb{P}^1(\mathbb{C}_p)\backslash A_i) = B_i^+$ and $\gamma_i(\mathbb{P}^1(\mathbb{C}_p)\backslash B_i^+) = A_i$. Then $\Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle \subset \mathrm{PGL}_2(\mathbb{Q}_p)$ is a p-adic Schottky group. Write $\Omega_\Gamma = \mathbb{P}^1 \backslash \Sigma_\Gamma$.

Theorem 14.4 (Mumford)**.** *If $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_p)$ is a p-adic Schottky group of rank g, then there exists a smooth projective totally split curve X_Γ of genus g over $\mathbb{Q}_p$ and a "p-adic analytic" uniformization isomorphism*

$$\Omega_\Gamma(\mathbb{C}_p)/\Gamma \xrightarrow{\sim} X_\Gamma(\mathbb{C}_p).$$

Furthermore, this uniformization is Galois equivariant, i.e., for any finite Galois extension $K/\mathbb{Q}_p$, the isomorphism

$$\Omega_\Gamma(K)/\Gamma \xrightarrow{\sim} X_\Gamma(K).$$

is equivariant under the natural action of the Galois group of $K/\mathbb{Q}_p$ on both sides.

Conversely, any smooth projective totally split curve X of genus g over $\mathbb{Q}_p$ admits a uniformization $\Omega_\Gamma(\mathbb{C}_p)/\Gamma \xrightarrow{\sim} X_\Gamma(\mathbb{C}_p)$ for some p-adic Schottky group $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_p)$ of rank g.

A curve X defined over a valued field K, with valuation ring R and residue field k, is *totally split* if X admits a proper flat model $\mathcal{X}$ over R whose special fiber $\mathcal{X}_k$ is a graph of $\mathbb{P}^1$s intersecting in k-rational points.

The Mumford uniformization works on the level of Berkovich analytic spaces and shows that the analytic space X^{an} of a smooth projective totally split curve X of genus g defined over $\mathbb{Q}_p$ is locally bianalytic to $(\mathbb{A}^1)^{\mathrm{an}}$ (actually bianalytic to an open subset of $(\mathbb{P}^1)^{\mathrm{an}}$). In fact, X^{an} is homotopy equivalent to a bouquet of g circles, so that $\pi_1(X^a n)$ is free of rank g. This is a way of understanding why the associated p-adic Schottky group is free.

14.3. **Mumford uniformization of the Jacobian.** Given a Mumford uniformization $\Omega_\Gamma(\mathbb{C}_p)/\Gamma \xrightarrow{\sim} X(\mathbb{C}_p)$ of a smooth projective totally split curve X of genus g over $\mathbb{Q}_p$, where $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_p)$ is a p-adic Schottky group, there is a very nice uniformization of the Jacobian of X.

Letting $H = \Gamma^{\mathrm{ab}}$ be the maximal abelian quotient, we have a canonical isomorphism $H \xrightarrow{\sim} H_1(X, \mathbb{Z})$. Choose free loxodromic generators $\Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle$ and their associated $q_1, \ldots, q_g \in \mathbb{C}_p^\times$ with $|q_i| < 1$. This defines an embedding of H into an affine algebraic torus.

Theorem 14.5 (Mumford)**.** *Let X be a smooth projective totally split curve of genus g over $\mathbb{Q}_p$ with Jacobian J. There is a p-adic analytic uniformization isomorphism*

$$(\mathbb{C}_p^\times)^g / \langle q_1, \ldots, q_g \rangle \xrightarrow{\sim} J(\mathbb{C}_p).$$

This generalizes the Tate uniformization of an elliptic curve.

14.4. **Some thoughts on uniformization in higher dimension.** Let's take a step back and consider why a theory of uniformization is useful, especially in higher dimension. In some sense, simply connected objects are easier to deal with than non-simply connected ones.

To understand a smooth projective variety X over $\mathbb{C}$ (or really any topological space), one can proceed in two steps:

(1) Understand the universal cover $\widetilde{X}$.
(2) Understand the action of the fundamental group $\pi_1(X)$ on $\widetilde{X}$.

The first example of this perspective is the uniformization theorem for Riemann surfaces reviewed above, which is especially nice since the universal covers have fixed geometry.

In a similar vein, consider abelian varieties. Every abelian variety of dimension g is $\mathbb{C}^g/\mathbb{Z}^{2g}$. So if we wanted to build a moduli space of abelian varieties, we can think of varying the abelian variety as varying the map $\mathbb{Z}^{2g} \to \mathrm{Aut}(\mathbb{C}^g)$. Though $\mathrm{Aut}(\mathbb{C}^g)$ is very complicated (and we do not even understand all regular automorphisms), we could instead consider the subgroup of affine transformations of $\mathbb{C}^g$, and then consider varying maps from $\mathbb{Z}^{2g}$. This gives a nice moduli space of abelian varieties. Note, however, that $\widetilde{X} \cong \mathbb{C}^g$ is only a biholomorphism; their algebraic structure is different.

Of course, this kind of program will not help much if X is already simply connected. Important examples include: X is rationally connected (e.g., $X \cong \mathbb{P}^2$), and X is a K3 surface.

If the fundamental group of X is finite, one can get some information from this program. For example, if X is an Enriques surfaces, then $\widetilde{X}$ is a K3 surface and the uniformization map is an étale double cover $\widetilde{X} \to X$, so $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$.

The universal cover of a product is the product of universal covers. So for example, if $X \cong$ Enriques surface $\times$ curve of genus 2 $\times$ Abelian variety, then $\widetilde{X}$ is biholomorphic to $\mathrm{K3} \times \mathbb{H} \times \mathbb{C}^g$.

Theorem 14.6 (Kollár–Pardon)**.** *If $\widetilde{X}$ is biholomorphic to a semialgebraic domain (i.e., cut out by norm inequalities on algebraic functions) in a quasiprojective variety, then*

$$\widetilde{X} \cong Y \times \mathbb{D}^r \times \mathbb{C}^s$$

where Y is quasi-projective and simply connected.

So an important open problem that arises is to classify all possible actions of a group $\pi_1(X)$ on a simply connected variety.

There is one more thing to keep in mind. If the universal cover is a projective variety, then by GAGA the covering map from the universal cover is a morphism of varieties. Hence the fundamental group is finite.

So the more interesting part is how $\pi_1(X)$ acts on the disk (hyperbolic space) or how it acts on affine space.

Thinking about Berkovich analytification as a version of complex analytic structure for varieties X over non-archimedian fields, we would like to understand the parallel story: What are the possibilities for $\widetilde{X}^{\mathrm{an}}$? How does $\pi_1(X^{\mathrm{an}})$ act on $\tilde{X}^{\mathrm{an}}$?

The case of Tate and Mumford uniformization of curves give us nice cases where this works, though only for certain classes of curves (i.e., totally split ones). Indeed, elliptic curves with good reduction have universal covers that are not isomorphic to algebraic varieties.

Beware! Every abelian variety of dimension g over $\mathbb{C}$ is isomorphic to $\mathbb{C}^g/\mathbb{Z}^{2g}$, but not conversely. There exist discrete subgroups $\mathbb{Z}^{2g}$ such that $\mathbb{C}^g/\mathbb{Z}^{2g}$ is not biholomorphic to any algebraic variety. Indeed, $\mathbb{Z}^{2g} \subset \mathbb{C}^g$ gives rise to an abelian variety iff the image of $\mathbb{Z}^{2g}$ satisfies the Hodge–Riemann bilinear relations.

The analogous problem for abelian varieties in the p-adic world still appears to be open: Characterize the subgroups $\mathbb{Z}^g \subset (\mathbb{C}_p^\times)^g$ such that the quotient is an abelian variety.

In the case of surfaces, there seem to be serious restrictions on the possible universal covers and fundamental groups allowable.

Theorem 14.7 (Cartwright)**.** *There does not exist any regular semistable surface X such that the dual complex Δ is homeomorphic to a surface of genus ≥ 2.*

However, there do exist regular semistable K3 surfaces X such that $\Delta \cong S^2$, i.e., Δ is a surface with genus 0. (In fact, for X a K3 surface, $X^{\mathrm{an}} \simeq S^2$ if and only if X^{an} is locally biholomorphic to $\mathbb{P}^2$. There is a similar story for abelian surfaces bianalytic to $S^1 \times S^1$.)

Here is an open problem: Show that there is no algebraic surface X such that X^{an} is locally bianalytic to $\mathbb{P}^2$ and also homotopy equivalent to an oriented surface of genus ≥ 2.

References

[Bak08] M. Baker. An introduction to Berkovich analytic spaces and non-Archimedean potential theory on curves. In *p-adic geometry*, volume 45 of *Univ. Lecture Ser.*, pages 123–174. Amer. Math. Soc., Providence, RI, 2008.

[Ber90] V. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[Car15] D. Cartwright. Excluded homeomorphism types for dual complexes of surfaces. preprint arXiv:1506.04712, 2015.

[GvdP80] L. Gerritzen and M. van der Put. *Schottky groups and Mumford curves*, volume 817 of *Lecture Notes in Mathematics*. Springer, 1980.

[KP12] J. Kollár and J. Pardon. Algebraic varieties with semialgebraic universal cover. *J. Topol.*, 5(1):199–212, 2012.

[Lü16] W. Lütkebohmert. *Rigid geometry of curves and their Jacobians*, volume 61 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, 2016.

[Mum72a] D. Mumford. An analytic construction of degenerating abelian varieties over complete rings. *Compositio Math.*, 24:239–272, 1972.

[Mum72b] D. Mumford. An analytic construction of degenerating curves over complete local rings. *Compositio Math.*, 24:129–174, 1972.

[Tat95] J. Tate. A review of non-Archimedean elliptic functions. In *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, Ser. Number Theory, I, pages 162–184. Int. Press, Cambridge, MA, 1995.