TROPICAL BRILL-NOETHER THEORY

15. The Jacobian of the skeleton is the skeleton of the Jacobian (after Baker and Rabinoff)

15.1. **Introductory remarks.** Throughout this note, we let K denote a field which is complete with respect to a non-trivial, non-Archimedean valuation, and we let R and k respectively denote the valuation ring and residue field of K. We use $\Lambda \subset \mathbb{R} \cup \{\infty\}$ to denote the value group of K.

Suppose we are given a smooth projective curve X over K and a semistable R-model $\mathfrak{X}$ of X. Let X^{an} denote the Berkovich analytification of X, and let G denote the dual graph of the special fiber of $\mathfrak{X}$. Recall from Lecture 11 that the regular realization Γ of G is a *skeleton* of G, in the sense that that associated to Γ there exist a natural inclusion $\Gamma \hookrightarrow X^{an}$ and a strong deformation retraction ρ from X^{an} onto the image of Γ.

The *Jacobian* J of X is a proper algebraic group parametrizing isomorphism classes of line bundles on X. Let J^{an} denote the analytification of J. Recall from [1, Chapter 6] that J^{an} has an associated *skeleton* $\Sigma(J^{an})$. Similarly to Γ, associated to $\Sigma(J^{an})$ there exist a natural inclusion $\Sigma(J^{an}) \hookrightarrow J^{an}$ and a strong deformation retraction from J^{an} onto the image of $\Sigma(J^{an})$.

Finally, recall from Lecture 9 that Γ, as a metric graph, has a associate tropical Jacobian $J(\Gamma)$ defined as the quotient $H_1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{R})$. Hence, each curve X has two associated real tori, namely $\Sigma(J^{an})$ and $J(\Gamma)$. In this note, we present the main result from [3]: for K algebraically closed, there is a canonical identification between $\Sigma(J^{an})$ and $J(\Gamma)$. In other words,

The skeleton of the Jacobian is the Jacobian of the skeleton.

15.2. **Note of Analytification.** Berkovich analytifications satisfy the following important property, which will be invoked throughout this note. Let X be a variety over K. Given a non-Archimedean extension L of K, there is a natural map $X(L) \to X^{an}$ defined as follow. Suppose we have a map Spec $L \to X$, corresponding to a point in $X(L)$. Let U be an affine open in X containing the image of Spec L.

Let $\|\cdot\|_L$ denote the non-Archimedean norm associated to L. We obtain a norm $\|\cdot\| \in U^{an} \subset X^{an}$ by composing $\|\cdot\|_L$ with the induced map $O_X(U) \to L$. Note that if L' is an extension of L, then the following commutes, where the map $X(L) \to X(L')$ is given by viewing an L point as an L' point.

$$\begin{tikzcd}
X(L) \arrow[r] \arrow[rr, bend left] & X(L') \arrow[r] & X^{an}
\end{tikzcd}$$

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Moreover, it is important to note for every point $x \in X^{an}$, there exists an extension L of K such that x lies in the image of $X(L) \to X^{an}$.

15.3. **Uniformization.** We now review the basics of uniformization of Jacobians. Recall that we have an exact sequence of K-analytic groups

$$0 \to M' \to E^{an} \to J^{an} \to 0$$

where E^{an} is the universal cover of J^{an}, and where M' is equal to the fundamental group $\pi_1(J^{an})$, realized as a discrete subgroup of the K-points $E^{an}(K)$ of E^{an}.

By lifting the analytic structure on J^{an}, we can view the universal cover E^{an} as K-analytic space. In fact, E^{an} is the Berkovich analytification of an algebraic group E, which fits into an exact sequence

$$0 \to T \to E \to B \to 0$$

where T is a split torus and B is an abelian variety with good reduction. These two exact sequences associated to J^{an} are usually presented as what is called a Raynaud cross:

$$\begin{tikzcd}
 & T^{\mathrm{an}} \arrow[d] & \\
M' \arrow[r] & E^{\mathrm{an}} \arrow[r] \arrow[d] & J^{\mathrm{an}} \\
 & B^{\mathrm{an}} &
\end{tikzcd}$$

The torus T is the generic fiber of a unique R-torus $\mathcal{T}$, and we will let $\overline{T}$ denote the special fiber of $\mathcal{T}$. Let T_0 denote the affinoid torus inside T^{an}, which is the locus of all points in T^{an} that has a well-defined specialization in $\overline{T}$. Then, there exists a unique analytic domain $E_0 \subset E^{an}$ and extension

$$0 \to T_0 \to E_0 \to B^{an} \to 0$$

such that we have the following commuting diagram.

$$\begin{tikzcd}
0 \arrow[r] & T_0 \arrow[r] \arrow[d, hook] & E_0 \arrow[r] \arrow[d] & B^{\mathrm{an}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T^{an} \arrow[r] & E^{\mathrm{an}} \arrow[r] & B^{\mathrm{an}} \arrow[r] & 0
\end{tikzcd}$$

Remark 15.1. In fact, let $\overline{B}$ denote the specialization of B. The analytic space E_0 is the unique compact analytic domain $E_0 \subset J^{an}$ whose special fiber $\overline{E}_0$ fits into the exact sequence

$$0 \to \overline{T}_0 \to \overline{E}_0 \to \overline{B} \to 0$$

given by specializing the exact sequence

$$0 \to T \to E \to B \to 0.$$

15.4. **The skeleton of J^{an}.** Let M denote the character lattice of T, and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the cocharacter lattice. The universal cover E^{an} also comes with a tropicalization map, which we now define, to the vector space $N_{\mathbb{R}} := N \otimes \mathbb{R}$. First, note that there is a map $\mathrm{trop} : T^{an} \twoheadrightarrow N_{\mathbb{R}}$, induced by the maps

$$\{T(L) \to N_{\mathbb{R}} : L \text{ is an extension of } K\}$$

given by coordinatewise valuation.

The fiber $\mathrm{trop}^{-1}(0)$ over 0 is precisely the domain T_0. Now, note that the diagram

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d, hook] & A_0 \arrow[d, hook] \\
T^{an} \arrow[r] & E^{an}
\end{tikzcd}$$

is in fact the push-out square with respect to the inclusions $T_0 \hookrightarrow E_0$ and $T_0 \hookrightarrow T^{an}$. In other words, if we have any other commuting diagram as follows,

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d, hook] & A_0 \arrow[d, hook] \\
T^{an} \arrow[r] & Y
\end{tikzcd}$$

then there is a unique induced map from $E^{an} \to Y$. In particular, this allows us to extend the map trop to a map $\mathrm{trop} : E^{an} \to N_{\mathbb{R}}$, by setting $\mathrm{trop}^{-1}(0) = E_0$.

The map trop is injective on the lattice M', viewed as a discrete subgroup of E^{an}, and its image $\mathrm{trop}(M')$ is a full-rank lattice in $N_{\mathbb{R}}$. Let Σ denote the quotient $N_{\mathbb{R}}/\,\mathrm{trop}(M')$. Since $J^{an} = E^{an}/M'$, the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\mathrm{trop}"] & E^{\mathrm{an}} \arrow[r] \arrow[d, "\mathrm{trop}"] & J^{\mathrm{an}} \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & \mathrm{trop}(M') \arrow[r] & N_{\mathbb{R}} \arrow[r] & \Sigma \arrow[r] & 0
\end{tikzcd}$$

where the map ρ is given by the universal property of quotients.

Theorem 15.2 ([1, Theorem 6.5.1])**.** *There is a section $N_{\mathbb{R}} \hookrightarrow E^{an}$, and a deformation retraction of E^{an} onto the image of $N_{\mathbb{R}}$. Moreover, the section and the deformation retraction are invariant under the action of M' on E^{an}, and therefore we have an induced section $\Sigma \hookrightarrow J^{an}$, and a deformation retraction of J^{an} onto the image of* Σ.

15.5. **Duality and Abel-Jacobi.** The quotient Σ is precisely the skeleton of J^{an}, first mentioned in the introductory section this note. Recall that our endgoal is construct a canonical identification between Σ and the tropical Jacobian $J(\Gamma)$.

First, we would like to convince the reader that, as a real torus, the dimension of Σ is equal to the dimension of $J(\Gamma)$, i.e. it is equal to the first Betti number of the dual metric graph Γ associated to X. Let $\check{J}$ denote the dual abelian variety associated to J. Recall that the dual $\check{J}$ is the the identity component $\mathrm{Pic}^0(J)$ of the Picard group of J.

Theorem 15.3 (Abel-Jacobi Theorem)**.** *There exists a canonical isomorphism* α^**, identifying* $\check{J}$ *with the Jacobian* $J = \mathrm{Pic}^0(X)$*.*

Let $\check{E}^{an}$ denote the universal cover of $\check{J}^{an}$, and just like for J we have an exact sequence

$$0 \to H_1(\check{J}^{an}, \mathbb{Z}) \to \check{E}^{an} \to \check{J}^{an} \to 0$$

where $H_1(\check{J}^{an}, \mathbb{Z})$ is realized as a discrete subgroup of $\check{E}^{an}(K)$. Bosch and Lütkebohmert [2, §6-7] have shown that isomorphism α^* induces an isomorphism between the lattice $H_1(\check{J}^{an}, \mathbb{Z})$ and the character lattice M of the torus T associated to J, inducing following commuting diagram.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\cong"] & \hat{E}^{\mathrm{an}} \arrow[r] \arrow[d] & \hat{J}^{\mathrm{an}} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & M' \arrow[r] & E^{\mathrm{an}} \arrow[r] & J^{\mathrm{an}} \arrow[r] & 0
\end{tikzcd}$$

Bosch and Lütkebohmert has also shown, using rigid analytic geometry, that there is a natural identification between M and the group $H_1(X^{an}, \mathbb{Z})$. In particular, since $H_1(X^{an}, \mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z})$, it follows that M' is lattice of rank equal to $b = \dim H_1(\Gamma, \mathbb{Z})$, and hence Σ is a real torus of dimension g.

15.6. **Tropicalizing divisors.** Thus, as topological spaces Σ and $J(\Gamma)$ are homeomorphic. Moreover, since $M \cong H_1(\Gamma, \mathbb{Z})$, the inclusion $M \hookrightarrow N_{\mathbb{R}}$ identify $N_{\mathbb{R}}$ with $H_1(\Gamma, \mathbb{R})$. Since $J(\Gamma)$ is equal to $H_1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z})$, we have an exact sequence

$$0 \to M \to N_{\mathbb{R}} \to J(\Gamma) \to 0.$$

We now construct a canonical homeomorphism $\Sigma \xrightarrow{\sim} J(\Gamma)$, identifying the map $r : J^{an} \to J(\Gamma)$ induced by retraction of divisors, with the retraction map $\rho : J^{an} \to \Sigma$.

We have a map $r : X(L) \to \Gamma$ obtain by composing the natural map $X(K) \to X^{an}$ with the retraction map $r : X^{an} \to \Gamma$. By extending linearly, this yields maps $r : \mathrm{Div}^d_K(X) \to \mathrm{Div}^d(\Gamma)$, where $\mathrm{Div}^d_K(X)$ denote the degree d divisors of X supported on $X(K)$. As noted in [3, Remark 5.4], the proposition below follows from the the non-Archimedean Poincaré-Lelong formula.

Proposition 15.4. *The maps* $\mathrm{Div}^0_K(X) \to \mathrm{Div}^0(\Gamma)$ *take principal divisors supported on* $X(K)$ *to principal divisors on* Γ*. In particular, we have an induced map* $r : J(K) \to J(\Gamma)$*.*

The retraction map $r : J(K) \to J(\Gamma)$ has the important property of being compatible with the quotient map $E(K) \to J(K)$.

Proposition 15.5 ([3, Proposition 5.3])**.** *The map* $r : J(K) \to J(\Gamma)$ *given by retraction of divisors is the unique map making the following square commutes*

$$\begin{tikzcd}
E(K) \arrow[r] \arrow[d, "\mathrm{trop}"] & J(K) \arrow[d, "r"] \\
N_{\mathbb{R}} \arrow[r] & J(\Gamma)
\end{tikzcd}$$

where $E(K) \to J(K)$ *is given by the map* $E^{an} \to J^{an}$*.*

Remark 15.6. The proof of the above theorem is highly technical. We refer the interested reader to the original paper of Baker and Rabinoff. Note however that the uniqueness of r is an easy consequence of the surjectivity of the left vertical arrow $E(K) \to J(K)$.

Now, notice that the diagram from the statement of the theorem extends to the following commuting diagram.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d] & E(K) \arrow[r] \arrow[d, "\mathrm{trop}"] & J(K) \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & M \arrow[r] & N_{\mathbb{R}} \arrow[r] & J(\Gamma) \arrow[r] & 0
\end{tikzcd}$$

Moreover, we have that $\mathrm{trop}(M') \subset M$, and hence we obtain an induced map

$$\pi : \Sigma = N_{\mathbb{R}}/\,\mathrm{trop}(M') \longrightarrow N_{\mathbb{R}}/M = J(\Gamma).$$

Corollary 15.7 ([3, Corollary 5.6])**.** *There exists a unique morphism r such that the following diagram commutes,*

$$\begin{tikzcd}
E^{\mathrm{an}} \arrow[r] \arrow[d, "\mathrm{trop}"] & J^{\mathrm{an}} \arrow[d, "r"] \\
N_{\mathbb{R}} \arrow[r] & J(\Gamma)
\end{tikzcd}$$

and whose restriction to $J(K)$ is precisely the retraction map $r : J(K) \to J(\Gamma)$.

Proof. Set $r = \pi \circ \rho$. The uniqueness of r follows from Proposition 15.5. □

15.7. **Proof of main theorem.** We now assume that the field K is algebraically closed. In particular, $X(K) \neq \emptyset$. Let P be an element of $X(K)$, and let p be the image of P under the map $r : X^{an} \to L$. Recall from Lecture 9 that we have a tropical Abel-Jacobi map $\alpha_p : \Gamma \to J(\Gamma)$. Let α_P denote the Abel-Jacobi map $\alpha_P : X^{an} \to J^{an}$.

Proposition 15.8 ([3, Proposition 6.1])**.** *The following diagram commutes:*

$$\begin{tikzcd}
X^{\mathrm{an}} \arrow[r, "\alpha_P"] \arrow[d] & J^{\mathrm{an}} \arrow[d, "r"] \\
\Gamma \arrow[r, "\alpha_p"] & J(\Gamma)
\end{tikzcd}$$

Proof. Consider the following diagram

$$\begin{tikzcd}
X(L) \arrow[r] \arrow[d] & \mathrm{Div}^0_L(X_L) \arrow[r] \arrow[d] & J(L) \arrow[d, "r"] \\
\Gamma \arrow[r] & \mathrm{Div}^0(\Gamma) \arrow[r] & J(\Gamma).
\end{tikzcd}$$

where the middle vertical arrows is given by extending linearly the left vertical arrow. By definition the leftmost square commutes, and the commutativity of rightmost square is a straightforward generalization of Proposition 15.4.

Now, we have the following diagram.

$$\begin{tikzcd}
X(L) \arrow[r] \arrow[d] & J(L) \arrow[d] \\
X^{\mathrm{an}} \arrow[r, "\alpha_P"] \arrow[d] & J^{\mathrm{an}} \arrow[d] \\
\Gamma \arrow[r, "\alpha_p"] & J(\Gamma)
\end{tikzcd}$$

Clearly the topmost square commutes. By our previous observations, the outer square commutes. Finally, note that every $x \in X^{\mathrm{an}}$ lifts to $X(L)$ for some extension L. Hence, we can derive the commutativity of bottom square by considering all extensions L of K. ☐

Recall that the induced map on homology

$$\alpha_{p,*} : H_1(\Gamma, \mathbb{Z}) \to H_1(J(\Gamma), \mathbb{Z})$$

is an isomorphism. The map $\alpha_{P,*} : H_1(X^{an}, \mathbb{Z}) \to H_1(J^{an}, \mathbb{Z})$ is also an isomorphism: $\alpha_{P,*}$ is precisely the isomorphism $M \cong M'$ induced by the Abel-Jacobi isomorphism $\hat{J} \cong J$. We can now prove the following.

Theorem 15.9 ([3, Proposition 2.9]). *The inclusion* $\operatorname{trop}(M') \subset M$ *is a bijection. In particular, the map* $\pi : \Sigma \to J(\Gamma)$ *is an isomorphism.*

Proof. By applying $H_1(\ , \mathbb{Z})$ to the diagram

$$\begin{tikzcd}
X^{\mathrm{an}} \arrow[r, "\alpha_P"] \arrow[d] & J^{\mathrm{an}} \arrow[d, "r"] \\
\Gamma \arrow[r, "\alpha_p"] & J(\Gamma)
\end{tikzcd}$$

we obtain the following diagram.

$$\begin{tikzcd}
H_1(X^{\mathrm{an}}, \mathbb{Z}) \arrow[r] \arrow[d] & H_1(J^{\mathrm{an}}, \mathbb{Z}) \arrow[r] \arrow[d, "r_*"] & E^{\mathrm{an}} \arrow[r] \arrow[d, "\mathrm{trop}"] & J^{\mathrm{an}} \arrow[d, "r"] \\
H_1(\Gamma, \mathbb{Z}) \arrow[r] & H_1(J(\Gamma), \mathbb{Z}) \arrow[r] & N_{\mathbb{R}} \arrow[r] & J(\Gamma)
\end{tikzcd}$$

We already know that the rightmost square commutes. The middle squares commutes because the map $E^{an} \to J^{an}$ and $N_{\mathbb{R}} \to J(\Gamma)$ are universal covering maps. Finally, the leftmost square commutes by functoriality of $H_1(\ , \mathbb{Z})$.

Now, the left vertical arrow of the leftmost square

$$\begin{tikzcd}
H_1(X^{\mathrm{an}}, \mathbb{Z}) \arrow[r] \arrow[d] & H_1(J^{\mathrm{an}}, \mathbb{Z}) \arrow[d, "r_*"] \\
H_1(\Gamma, \mathbb{Z}) \arrow[r] & H_1(J(\Gamma), \mathbb{Z})
\end{tikzcd}$$

is an isomorphism, as the three other arrows are isomorphisms. In particular, it follows that the image of $M' = H_1(J^{\mathrm{an}}, \mathbb{Z})$ under trop is equal to $M = H_1(J(\Gamma), \mathbb{Z})$, which is the desired statement. □

References

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