TROPICAL BRILL-NOETHER THEORY

16. Lifting tropical intersections

For a number of applications of tropical techniques to classical algebro-geometric questions, one requires so called "lifting theorems". For instance, given a special divisor D on a metric graph Γ, with prescribed rank r and degree d, it is often useful to know if there exists a curve $\mathcal{C}$ over a non-archimedean field whose skeleton is Γ, and a divisor D' of rank r and degree d on $\mathcal{C}$ that specializes to D. A fruitful technique to answer such questions is to study intersection theory on Jacobians of graphs and curves. By Raynaud's uniformization theorems, this can further be reduced to questions about intersections of cycles in tori and their tropicalization. This will be the topic of the present lecture. We will explain Rabinoff's lifting theorem for top intersections of hypersurfaces in tori. The lecture that follows will concern lifting divisors on the chain of loops, where these techniques are heavily employed.

Let K be a complete non-archimedean field and $M \cong \mathbb{Z}^n$ a lattice. Denote by $T = \operatorname{Spec}(K[M])$ the algebraic torus with character lattice M, and by T^{an} the Berkovich analytification. Recall that each point $x \in T^{\mathrm{an}}$ is a valuation on the coordinate ring of T:

$$\mathrm{val}_x : K[M] \to \mathbb{R} \cup \{\infty\}.$$

By restricting val_x to the character lattice of T we obtain a map

$$M \to \mathbb{R} \cup \{\infty\}.$$

Since the elements of M are all invertible functions, the image of this map lies in $\mathbb{R}$, and we obtain a map of abelian groups $M \to \mathbb{R}$. In other words, restriction to characters produces a well-defined, continuous, *tropicalization* map

$$T^{\mathrm{an}} \to N_{\mathbb{R}} := \operatorname{Hom}(M, \mathbb{R}).$$

Continuity follows immediately from the definition of the Berkovich topology.

Let $f_1, \ldots, f_n \in K[M]$ be non-monomial Laurent polynomials and $V(f_i) \subset T$ the hypersurfaces that they cut out. Define $\operatorname{trop}(f_i) = \operatorname{trop}(V(f_i))$.

The following is a consequence of a result of Bieri and Groves.

Fact (Bieri-Groves Theorem): The set $\operatorname{trop}(f_i)$ can be given the structure of a polyhedral complex of pure dimension $n - 1$.

Example: Suppose $M = \mathbb{Z}^2$ generated multiplicatively by monomials x and y, and let $f = x + y + 1$. The hypersurface $V(f) \subset (K^*)^2$ is a generic line in $\mathbb{P}^2$ intersected with its dense torus. Its tropicalization is depicted in the figure below.

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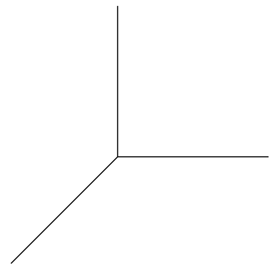

FIGURE 1. The tropicalization of a generic line in $(K^*)^2$.

Given a Laurent polynomial $f = \sum_{u \in M} a_u x^u$, one may construct the *Newton polytope* $\mathrm{Newt}(f)$ *of* f, and a polyhedral complex called the *Newton complex*, a particular subdivision of $\mathrm{Newt}(f)$. These are defined as follows. The Newton polytope is defined as the convex hull of those monomials in f appearing with nonzero coefficient:

$$\mathrm{Newt}(f) = \mathrm{conv}\{u \in M : a_u \neq 0\}.$$

To construct the Newton complex, we consider the set of points $S = \{(u, \nu(a_u)) : \nu(a_u) < \infty\}$ as a subset of $M \times \mathbb{R}$. The projection to M of the lower convex hull of this set induces a subdivision of $\mathrm{Newt}(f)$ which is the Newton complex $\widetilde{\mathrm{Newt}(f)}$. When K has the trivial valuation, the Newton complex and the Newton polytope coincide.

Given Laurent polynomials $f_1, \ldots, f_n$, Rabinoff's lifting theorem compares the intersection $\bigcap_{i=1}^{n} \mathrm{trop}(f_i)$ and the scheme theoretic intersection $\bigcap_{i=1}^{n} V(f_i)$. We henceforth assume that the latter is a finite number of possibly non-reduced points. This interplay between these intersection theory is rich, and is closely related to the theory of Minkowski sums and mixed volumes of polytopes. We require the following definition.

Given a polytope P and $\lambda \in \mathbb{R}_{>0}$, let λP be the dilation of P by λ. Given polytopes P_1 and P_2, let $P_1 + P_2$ be the Minkowski sum.

Definition 16.1 (Mixed Volume). *Let $\underline{P} = (P_1, \ldots, P_n)$ with P_i polytopes in $\mathbb{R}^n$. Define $V_{\underline{P}}(\lambda_1, \ldots, \lambda_n) = \mathrm{vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ (scaling and then Minkowski sum) for $\lambda_i \in \mathbb{R}_{\geq 0}$. $V_{\underline{P}}$ is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n$ of degree n.*
The mixed volume of $P_1, \ldots, P_n$, denoted $MV(\underline{P}) = MV(P_1, \ldots, P_n)$ is the coefficient of $\lambda_1 \cdots \lambda_n$ in this polynomial.

In what follows, we assume that the reader is familiar with the basics of the theory of toric varieties. We very briefly recall that a toric variety is a normal equivariant partial compactification of T. More precisely, it is a normal variety X containing T as a dense open, together with an action of T on X that extends the natural action of T on itself. Examples include $\mathbb{P}^n, \mathbb{P}^n \times \mathbb{P}^m$, and blowups of such along T-invariant subvarieties.

Toric varieties can be recovered from purely polyhedral data known as *fans* in $N_{\mathbb{R}}$. A *cone* in $N_{\mathbb{R}}$ is a finite intersection of integral half-spaces containing the origin and containing no linear subspace of $N_{\mathbb{R}}$. Every cone σ can be recovered as the space of monoid homomorphisms, $\mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0})$ where S_σ is the monoid of functions in M that are nonnegative on σ. The toric variety associated to σ is the spectrum of the the associated monoid algebra. That is, $U_\sigma = \mathrm{Spec}(K[S_\sigma])$. A *fan* is a collection of cones in $N_{\mathbb{R}}$ such that the intersection of any two cones is a face of each. A fan encodes a manner in which to glue the affine toric varieties associated to cones to obtain a global toric variety. We refer the reader to Chapters 1 and 2 of Fulton's text on toric varieties for details.

Let Δ be a fan in $N_{\mathbb{R}}$. Tropicalization extends naturally to the a continuous map from the analytic toric variety $X(\Delta)^{\mathrm{an}}$. That is, we have the commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & N_{\mathbb{R}} \arrow[d] \\ X(\Delta)^{\mathrm{an}} \arrow[r] & N(\Delta). \end{tikzcd}$$

Given $f \in K[M]$, the normal fan to $\mathrm{Newt}(f)$ is denoted $\Delta(f)$. This coincides with the recession fan, i.e. the fan of cones of unbounded directions, of the polyhedral complex $\mathrm{trop}(f)$.

Just as scheme theoretic intersections must be counted with multiplicities, tropical intersections must also be counted with multiplicities. It is at this stage that mixed volumes play a key role. We will define tropical multiplicities in two steps, dealing first with the case of an isolated intersection point, and then perturbing the intersection to reduce the general case to this.

Definition 16.2 (Tropical Multiplicities, Part I). *Let $v \in \bigcap_{i=1}^{n} \mathrm{trop}(f_i)$ be an isolated point. The **stable tropical intersection multiplicity** at v of $\{\mathrm{trop}(f_i)\}_{i=1}^n$ is*

$$i(v, \mathrm{trop}(f_1), \ldots, \mathrm{trop}(f_n)) := MV(\gamma_1, \ldots, \gamma_n)$$

where γ_i is the polytope corresponding to v in the Newton complex of f_i.

To define tropical multiplicities in general, we need the following "perturbation" of a polytope.

Definition 16.3 (Thickening). *Let P be a polytope, expressed as a a finite intersection of halfspaces:*

$$P = \bigcap_{i=1}^{r} \{v \in N_{\mathbb{R}} | \langle u_i, v\rangle \leq a_i\}$$

*with fixed u_i and a_i. A **thickening** of P is any polytope of the form*

$$P' = \bigcap_{i=1}^{r} \{v \in N_{\mathbb{R}} | \langle u_i, v\rangle \leq a_i + \varepsilon\}$$

for $\varepsilon > 0$.

In our definition of tropical multiplicities, it could be that all the complexes $\mathrm{trop}(f_i)$ share a face. A point in such a face would not be isolated.

Definition 16.4 (Tropical Multiplicities, Part 2). *Let $C \subset \bigcap \operatorname{trop}(f_i)$ be a connected component of the intersection. Choose integral vectors $v_1, \ldots, v_n$, $\varepsilon > 0$ and P a thickening of the complex underlying C, such that*

$$|P| \cap \bigcap_{i=1}^{n} (\operatorname{trop}(f_i) + \varepsilon v_i) \tag{1}$$

is finite. Define

$$i(C, \operatorname{trop}(f_1), \ldots, \operatorname{trop}(f_n)) = \sum i(v, \operatorname{trop}(f_1) + \varepsilon v_1, \ldots, \operatorname{trop}(f_n) + \varepsilon v_n)$$

where the sum is over all points in the intersection (1).

Implicit in this definition is the fact that such a multiplicity is well-defined – i.e. that a thickening always achieves an isolated intersection and that the intersection multiplicity is independent of the thickening.

We have arrived at Rabinoff's lifting theorem, which informally states that, counting components of the intersection with the multiplicities as described above, the classical intersection number of hypersurfaces $V(f_i)$ can be read off the intersection of their tropicalizations. In order to not lose intersection points "at infinity", we need to work in an appropriate compactification of T.

Theorem 16.5 (Rabinoff's Theorem). *Let $f_1, \ldots, f_n \in K[M]$ be nonzero. Define the fan Δ to be the intersection $\bigcap_{i=1}^{n} \Delta(f_i)$. Assume Δ is pointed. Let $C \subset \bigcap_{i=1}^{n} \operatorname{trop}(f_i)$ be a connected component. Let $\overline{C}$ be the closure of C in $N(\Delta)$. Let $Y_i = \overline{V(f_i)}$ in $X(\Delta)$, and $Y = \bigcap Y_i$. Then*

$$i(c, \operatorname{trop}(f_1), \ldots, \operatorname{trop}(f_n)) = \sum_{\substack{\operatorname{trop}(\xi) \in \overline{C} \\ \xi \in Y}} \dim_K(\mathcal{O}_{Y,\xi}).$$

References

[Rab12] J. Rabinoff. Tropical analytic geometry, Newton polygons, and tropical intersections. *Adv. Math.*, 229(6):3192–3255, 2012.