MATH 665: TROPICAL BRILL-NOETHER THEORY

2. Reduced divisors

The main topic for today is the theory of v-reduced divisors, which are canonical representatives of divisor classes on graphs, depending only on the choice of a base vertex v. We will prove the existence and uniqueness of v-reduced divisors, along with some of the fundamental properties that make them essential tools in this subject. Reduced divisors will play prominent roles in the proof of tropical Riemann–Roch and in the characterization of special divisors on a generic chain of loops (the key result about divisors on graphs underlying the tropical proof of the Brill–Noether theorem).

References for today's lecture include Matt Baker's blog post about reduced divisors and Riemann–Roch (2014), the paper of Hladký–Král–Norine (2013), and the original paper of Baker–Norine (2007).

Let G be a finite graph. The group of divisors on G is the free abelian group on the set of vertices

$$\mathrm{Div}(G) = \mathbb{Z}^{V(G)}.$$

We think of the graph G as being analogous to a smooth projective algebraic curve, with the image of the combinatorial Laplacian $\Delta(G)$ playing the role of the principal divisors. The Picard group (or divisor class group) is then

$$\mathrm{Pic}(G) = \mathrm{Div}(G)/\operatorname{Im}(\Delta),$$

which is graded by degree, since the image of $\Delta(G)$ is contained in the subgroup of divisors of degree zero. The Jacobian of the graph is

$$\mathrm{Jac}(G) = \mathrm{Pic}^0(G),$$

the group of divisor classes of degree zero modulo the image of $\Delta(G)$. Note that the Jacobian of a smooth projective algebraic curve is a compact abelian group. The Jacobian of a graph is also a compact abelian group—it is abelian and finite.

We say that a divisor $D = \sum a_i v_i$ is *effective* if all of the coefficients a_i are nonnegative, and a divisor class $[D] \in \mathrm{Pic}(G)$ is effective if it contains an effective representative.

Recall that the (Baker–Norine) rank $r(D)$ is the largest integer r such that $[D-E]$ is effective for all effective divisors E of degree r, and has the following properties:

- the rank $r(D)$ depends only on $[D]$.
- if $[D]$ is not effective then $r(D) = -1$.
- if D is effective then $r(D) \geq 0$.

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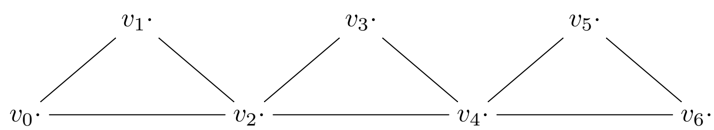
Note that every graph has divisors of arbitrarily large rank. For instance,

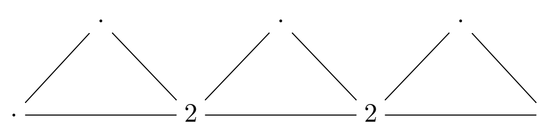
$$r\big(b \sum_{v \in V(G)} v\big) \geq b.$$

Example 2.1. Here we consider a chain of three triangles.



From last time, we know that the Jacobian of a graph obtained by gluing two graphs along a vertex is the sum of the Jacobians of the original graph, and the Jacobian of a triangle is $\mathbb{Z}/3\mathbb{Z}$. Applying that lemma twice shows that $\mathrm{Jac}(G) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 3}$. One can also compute directly that the number of spanning trees is 27.

We consider the divisor $D = 2v_2 + 2v_4$, i.e. twice the sum of the two cut vertices.



We claim that $r(D) = 2$. On the blackboard, we did an explicit sequence of chip-firings to check that $[D - E]$ is effective for every effective E of degree 2, up to symmetry. This shows that $r(D)$ is at least 2. Explicit chip-firings also show that $[D - 3v_i]$ is effective for all i. However, $[D - 2v_0 - v_5]$ is not effective, and we will check this by the end of class, using v-reduced divisors.

(Next week, we will see that D is the *canonical divisor* K_G on this graph, and the fact that $r(D) = 2$ follows trivially from Riemann–Roch for graphs!)

In small cases, such as the example above, we can check by hand that a rank is at least some number by considering cases and saying how many chips to fire from where. But we don't currently have a good way of giving nontrivial upper bounds for the rank. We could do something brute force, using the image of $\Delta(G)$ to compute all effective representatives of $[D - E]$ for all effective E of given degree r, but that would be painful.

The key technique for giving nontrivial upper bounds on the rank is the theory of v-reduced divisors. These are canonical representatives of divisor classes in $\mathrm{Pic}(G)$, depending only on the choice of a base vertex v.

We introduce a couple more pieces of notation, which will be useful for describing explicit equivalences of divisors. Let f be an element of $\mathbb{Z}^{V(G)}$ considered as the domain of Δ. So we are thinking of f as the analogue of a rational function, rather than as a divisor. (Note that there is a unique piecewise linear function on the geometric realization of G, that takes the values given by f on the vertices and is linear (with integer slope) along each edge; when we make the transition from discrete graphs to metric graphs. This function will play the role of f, when we make the transition from discrete graphs to metric graphs.)

We write D_f for $\Delta(f) \in \mathrm{Div}(G)$. Also, for $A \subset V(G)$, we let $D_A = D_{\chi(A)}$. In other words, adding D_A to a divisor is the same as "firing all vertices of A at once".

Fix a vertex v in G. We will say that a divisor $V = \sum a_i v_i$ is effective away from v if $a_i \geq 0$ for $v_i \neq v$. Similarly we can talk about divisors effective away from A for any $A \subset V(G)$.

Observe that if D is effective away from v then $D + D_A$ is effective away from $\{v\} \cup A$, and $D + D_A$ is effective at $v_i \in A$ if and only if $\mathrm{outdeg}_{v_i}(A) \leq a_i$. Here, $\mathrm{outdeg}_{v_i}(A)$ is the number of edges incident on v_i that connect to a vertex not in A.

Definition 2.2. *A divisor D is* v-reduced *if it is effective away from v, and for any subset $A \subset V(G)\backslash\{v\}$, $D + D_A$ is not effective away from v.*

As a first step toward proving that every divisor class has a unique v-reduced representative, we prove the following lemma. (Note that there are multiple different ways of proving this lemma, and everything else that we are doing today—we will discuss some of the alternate perspectives later in class.)

Lemma 2.3. *Every divisor class contains a representative that is effective away from v.*

Proof. The group $\mathrm{Jac}(G)$ is finite, so $m[v_i - v]$ is trivial for some $m > 0$. Therefore, starting from any representative of a given divisor class, we can add an appropriate positive integer multiple of $m[v_i - v]$ to arrive at a representative that is effective away from v. □

Proposition 2.4. *Every divisor class contains a unique v-reduced divisor.*

Proof. (Existence) Order the vertices so that every vertex other than v has a neighbor that precedes it. (To do this, let v be the first vertex in the order and then take the neighbors of v in any order, followed by their neighbors in any order, and so on.)

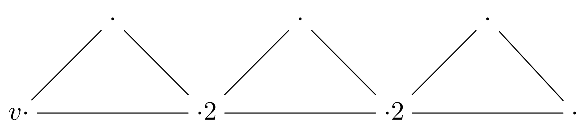
Start with an arbitrary class in $\mathrm{Pic}(G)$ and choose a representative D that is effective away from v. Either D is v-reduced or there is a subset $A \subset V(G)\backslash\{v\}$ such that $D + D_A$ is effective away from v. Then because A is not the whole graph it has some neighbor that precedes it, so $D + D_A > D$ in the lexicographic ordering. (Idea: We order the divisors by the lexicographic ordering from the ordering of the vertices, then firing off all of A gives a chip to a neighbor of something in A such that the neighbor precedes all of A.) Now replace D by $D + D_A$ and repeat. This procedure terminates because there are only finitely many divisors of a given degree that are effective away from v and greater than a given divisor.

(Uniqueness) Suppose $D \sim D'$ are distinct divisors, both effective away from v. We show that only one of D, D' can be v-reduced. We have $D' - D = D_f$ with f non-constant, and we may assume that f achieves its maximum on a set $A \subset V(G) \backslash \{v\}$ (by interchanging D, D' if necessary). Write $D' + D_A = b_0 v_0 + \cdots + b_s v_s$, $D = \sum_{i=0}^{s} a_i v_i$, $D' = \sum_{i=0}^{s} a'_i v_i$. Then $b_i = a'_i - \text{outdeg}_{v_i}(A) \geq a'_i - \sum_{v_i z \in E(G)} (f(v_i) - f(z)) = a_i$ for $v_i \in A$ and $b_j \geq a'_j$ for $v_j \notin A$. So $D' + D_A$ is effective away from v. Thus D' is not v_0-reduced. $\square$

Other proofs of existence and uniqueness of v-reduced divisors involve G-parking functions (Postnikov) or the stable recurrent configurations in the abelian sandpile model. See, for instance, the survey article by Holroyd, Levine, et.al. "Chip firing and rotor-routing on directed graphs."

Our proof of existence and uniqueness implicitly gives the following efficient algorithm for finding the v-reduced divisor equivalent to a given divisor.

Dhar's Burning Algorithm:



Given a divisor D and a vertex v, we may compute the v-reduced divisor equivalent to D as follows:

(1) Replace D with an equivalent divisor that is effective away from v, using the technique of Lemma 2.3 above.
(2) Start a fire at v.
(3) Burn every edge of the graph that is adjacent to a burnt vertex.
(4) Running through the vertices in lexicographic order, burn the first vertex w with the property that the number of burnt edges adjacent to w exceeds the number of chips $D(w)$ at w. If no such vertex exists, proceed to step (5). Otherwise, return to step (3).
(5) Let A be the set of unburnt vertices. If A is non-empty, then $D + D_A$ is effective away from v. Replace D with $D + D_A$ and return to step (2). Otherwise, if A is empty, then D is v-reduced.

To see that the algorithm works, note that by construction $D(w) \geq \text{outdeg}_w(A)$ for all $w \in A$. As in the proof of Proposition 2.4, it follows that $D + D_A$ is effective away from v. Conversely, if A' is a set of vertices such that $D + D_{A'}$ is effective away from v for some $A' \subset V(G) \backslash \{v\}$, then A' consist only of unburnt vertices. That is, the set of unburnt vertices is lexicographically minimal amongst all sets A' such that $D + D_{A'}$ is effective away from v. To see this, let $w \in A'$ be the first vertex in A' that burns. By construction, this implies that $D(w) < \text{outdeg}_w(A')$, hence $D + D_{A'}$ is not effective away from v. The algorithm terminates for the same reason as in Proposition 2.4 – there are only finitely many divisors of given degree that are greater than a given divisor in the lexicographic ordering.

Question 2.5. *Compute the running time of Dhar's burning algorithm, as a function of $\#E(G)$, $\#V(G)$, and $\deg(D)$. What is the optimal algorithm for finding v-reduced divisors in a given class?*

Remark 2.6. A divisor D is v-reduced if and only if $D + v$ is v-reduced. So it is enough to compute v-reduced representatives for every element of $\mathrm{Jac}(G)$ to have v-reduced representatives for every element of $\mathrm{Pic}(G)$.

Dhar's burning algorithm gives the following characterization of effective divisor classes, in terms of their v-reduced representatives. *For the purposes of Riemann–Roch and Brill–Noether theory, this characterization of effectiveness is the most important property of v-reduced divisors.*

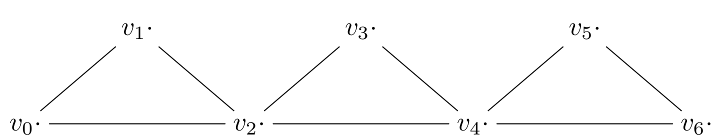
Proposition 2.7. *The divisor class $[D]$ is effective if and only if its v-reduced representative is effective.*

Proof. If the v-reduced representative is effective, then the class effective. The converse follows from Dhar's burning algorithm; if we start with an effective representative of the divisor class and run Dhar's algorithm, then the divisor stays effective at every step. □

As a consequence of this proposition, we can check whether a divisor has rank$\geq r$ by running Dhar's burning algorithm on $D - E$ for each of the $O(\#V(G))^r$ effective divisors E of degree r. Together with the Riemann–Roch theorem, which determines the rank of any divisor of degree greater than $2g - 2$, this gives an efficient algorithm for computing $r(D)$. However, determining the smallest degree of a divisor of a given rank, e.g. the *gonality*, which is the smallest degree of a divisor of positive rank, is NP hard. See (Gijswijt 2015).

We now return to our example from the beginning of class, and complete the proof that the rank of the divisor in question is exactly 2.

Example 2.8. Recall that we are working with the chain of three triangles



We claim that an arbitrary divisor $D = a_0v_0 + \cdots + a_6v_6$ is v_0-reduced if and only if $a_i \geq 0$ for $i \geq 1$ and $a_1 + a_2 \leq 1$, $a_3 + a_3 \leq 1$, $a_5 + a_6 \leq 1$. It is easy to check one direction, that each of these divisors is v_0-reduced. Then one can count that there are 27 such divisors in each degree, and hence these are all of the v_0-reduced divisors.

To complete the proof that $r(2v_2 + 2v_4) = 2$, it remains to show that $[-2v_0 + 2v_2 + 2v_4 - v_5]$ is not effective. To see this, note that $-v_0 + v_1 + v_6$ is in this class and v_0-reduced, and apply the preceding proposition.

Exercise 2.9. Classify the v_2-reduced and v_3-reduced divisors on a chain of three triangles.

References

[Bak13] M. Baker. Riemann-Roch for graphs and applications. Matt Baker's Math Blog, 2013. URL:https://mattbaker.blog/2013/10/18/riemann-roch-for-graphs-and-applications/.

[Bak14] M. Baker. Reduced divisors and Riemann-Roch for graphs. Matt Baker's Math Blog, 2014. URL:https://mattbaker.blog/2014/01/12/reduced-divisors-and-riemann-roch-for-graphs/.

[BN07] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.

[HKN13] J. Hladký, D. Kráľ, and S. Norine. Rank of divisors on tropical curves. *J. Combin. Theory Ser. A*, 120(7):1521–1538, 2013.