TROPICAL BRILL–NOETHER THEORY

3. Riemann–Roch for Graphs

The main references for today's talk are the blog post of Matt Baker (2014) and the original paper of Baker and Norine (2007).

Fix a connected graph G.

Definition 3.1. *The* genus *of G is $g = |E| - |V| + 1$.*

If one views G as a 1-dimensional complex, then the genus is the dimension of $H_1(G)$. One can see this by contracting a spanning tree of G to a point, leaving a wedge of g loops.

Remark 3.2. The genus considered here is distinct from the familiar "topological genus" of a graph, which is defined as the least g such that G can be embedded on a closed oriented surface of genus g and hard to compute. Note, however, that if G is embedded on a closed surface of Euler characteristic χ with each face simply connected, then $g = f - \chi + 1$, where f is the number of faces in the embedding.

Definition 3.3. *The* canonical divisor $K \in \operatorname{Div} G$ *is* $K = \sum_v (\operatorname{val} v - 2)v$.

Observe that $\deg(K) = 2g - 2$. In other words, $\deg K = 2|E| - 2|V|$, which is clear because each edge contributes to the coefficients of 2 vertices, and each vertex contributes -2.

We can now state the Riemann-Roch theorem for graphs.

Theorem 3.4 (Riemann-Roch for Graphs)**.** *For every $D \in \operatorname{Div} G$,*

$$\operatorname{rk}(D) - \operatorname{rk}(K - D) = \deg(D) - g + 1.$$

This has exactly the same form as the Riemann-Roch theorem for Riemann surfaces, but the rank and the genus have been defined differently.

Corollary 3.5. $\operatorname{rk}(D) \geq \deg(D) - g$, *with equality if* $\deg(D) > 2g - 2$.

There are many proofs of the Riemann-Roch theorem. The proof we will present is a simplification of Baker and Norine's original proof, presented on Baker's blog.

Definition 3.6. *An* orientation $\mathcal{O}$ *on G is a directed graph with underlying graph G. We call $\mathcal{O}$ an* acyclic orientation *if the directed graph contains no directed cycles. For any orientation $\mathcal{O}$, we let $D_{\mathcal{O}} = \sum_v (\operatorname{indeg} v - 1)v$ be the* orientation divisor *corresponding to $\mathcal{O}$.*

The dual orientation, written $\tilde{\mathcal{O}}$, is the orientation given by reversing all of the edges in $\mathcal{O}$.

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Note that $D_{\mathcal{O}} + D_{\bar{\mathcal{O}}} = K$.

Lemma 3.7. *For any acyclic orientation $\mathcal{O}$, The divisor class $[D_{\mathcal{O}}]$ is not effective,*

Proof. Suppose $D \in [D_{\mathcal{O}}]$, and write $D = D_{\mathcal{O}} + D_f$, and let S be the set of vertices where f is maximal. Since $\mathcal{O}$ is acyclic, we may choose v to be initial in S, so $\mathrm{indeg}_S v = 0$.

Then $D(v) < 0$. Indeed, begin with $D_{\mathcal{O}}$ and perform the sequence of vertex firings corresponding to adding D_f. Then v loses at least one chip for each edge from the complement of S to v. But the number of such edges is $\mathrm{indeg}_{V\setminus S} v = \mathrm{indeg} v = D_{\mathcal{O}}(v) + 1$. So D is not effective. $\square$

Lemma 3.8. *For all $D \in \operatorname{Div} G$, either $[D]$ is effective or there is an acyclic orientation $\mathcal{O}$ such that $[D_{\mathcal{O}} - D]$ is effective (but not both).*

Proof. Given $D \in \operatorname{Div} G$, we may assume that D is q-reduced for some vertex q. Run Dhar's algorithm, and orient each edge in the direction it burns. Because D is q-reduced, every edge burns, so this gives an orientation $\mathcal{O}$ on G. To see that $\mathcal{O}$ is acyclic, note that every time a vertex burns, all of the adjacent edges burn, hence one cannot proceed via a directed path from a newly burnt to a previously burnt vertex.

For all $v \neq q$, $D(v) \leq D_{\mathcal{O}}(v)$ because otherwise the fire would not have reached v. If $D(q) \geq 0$, then D is effective. Otherwise, $D(q) \leq -1 = \mathrm{indeg}_{\mathcal{O}} q - 1 = D_{\mathcal{O}}(q)$, so $D \leq D_{\mathcal{O}}$ everywhere. So $D_{\mathcal{O}} - D$ is effective. $\square$

For $D \in \operatorname{Div} G$, we can write $D = D^+ - D^-$ where D^+ and D^- are effective and have disjoint supports. A moment's thought reveals that this decomposition is unique. We write $\deg^+(D) = \deg(D)^+ = \sum_{v : D(v) > 0} D(v)$.

Lemma 3.9. *For any $D \in \operatorname{Div} G$,*

$$\mathrm{rk}(D) = \min_{\substack{D' \in [D] \\ \mathcal{O}}} \deg^+(D' - D_{\mathcal{O}}) - 1.$$

Proof. ($\leq$) Take $D^{\min}, \mathcal{O}$ achieving this minimum. Write $D^{\min} - D_{\mathcal{O}} = E^+ - E^-$ with

$$\deg(E^+) = \deg^+(D^{\min} - D_{\mathcal{O}}) = \min_{\substack{D' \in [D] \\ \mathcal{O}}} \deg^+(D' - D_{\mathcal{O}}).$$

Rearranging, we have $[D - E^+] = [D_{\mathcal{O}} - E^-]$, which is not effective by Lemma 3.7. So

$$\mathrm{rk}(D) < \deg(E)^+ = \min_{\substack{D' \in [D] \\ \mathcal{O}}} \deg^+(D' - D_{\mathcal{O}}).$$

($\geq$) Suppose to the contrary that

$$\mathrm{rk}(D) < \min_{\substack{D' \in [D] \\ \mathcal{O}}} \deg^+(D' - D_{\mathcal{O}}) - 1.$$

Then there exists E effective with

$$\deg(E) = \min_{\substack{D' \in [D] \\ \mathcal{O}}} \deg^+(D' - D_{\mathcal{O}}) - 1$$

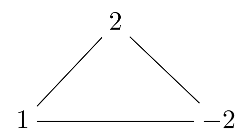
and $[D-E]$ not effective. By the Lemma 3.8, there exists $\mathcal{O}$ such that $[D_{\mathcal{O}}-(D-E)]$ is effective, say $D''-(D-E)=E'$ for some $D''\in[D]$ and E' effective. Rearranging this, $D''-D_{\mathcal{O}}=E-E'$, so

$$\deg^+(D''-D_{\mathcal{O}})=\deg^+(E-E')\leq\deg^+(E)=\deg(E)=\min_{\substack{D'\in[D]\\ \mathcal{O}}}\deg^+(D'-D_{\mathcal{O}})-1,$$

a contradiction. □

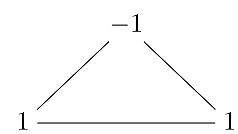
Example 3.10. On the triangle graph, the divisor

$D=$



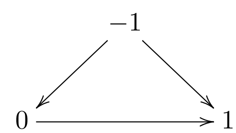
has rank 0, as can easily be verified. It is equivalent to the divisor

$D'=$.



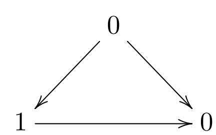
There is an acyclic orientation O with

$D_O=$



which gives

$D'-D_O=$



with $\deg^+(D-D_O)-1=0$.

Summarizing what we have so far, we have a way to express $\mathrm{rk}(D)$ as a minimum over orientations. The way K will come into this is that $\mathcal{O}\mapsto\tilde{\mathcal{O}}$ gives an involution on orientations with $D_{\mathcal{O}}+D_{\tilde{\mathcal{O}}}=K$.

Proof of Riemann-Roch theorem. We make two observations:

(1) $D_{\mathcal{O}}+D_{\tilde{\mathcal{O}}}=K$. So

$$\min_{\substack{D'\in[D]\\ \mathcal{O}}}\deg^+(D'-D_{\mathcal{O}})=\min_{\substack{D'\in[D]\\ \mathcal{O}}}\deg^+(D'-(K-D_{\mathcal{O}})).$$

(2) $\deg^+(D)=\deg(D)+\deg^+(-D)$. So

$$\deg^+(D-D_{\mathcal{O}})=\deg(D-D_{\mathcal{O}})+\deg^+(D_{\mathcal{O}}-D)=\deg(D)+\deg^+(D_{\mathcal{O}}-D)-(g-1).$$

Now we have

$$\begin{aligned}
\mathrm{rk}(D) - \mathrm{rk}(K-D) &= \min_{\substack{D'\in[D]\\ \mathcal{O}}} \deg^+(D' - D_{\mathcal{O}}) - \min_{\substack{D'\in\\ \mathcal{O}}} \deg^+(K - D' - D_{\mathcal{O}}) \\
&= \min_{\substack{D'\in[D]\\ \mathcal{O}}} \deg^+(D' - D_{\mathcal{O}}) - \min_{\substack{D'\in[D]\\ \mathcal{O}}} \deg^+(D_{\mathcal{O}} - D') \\
&= \min_{\substack{D'\in[D]\\ \mathcal{O}}} (\deg(D') + \deg^+(D_{\mathcal{O}} - D') - (g-1)) - \min_{\substack{D'\in[D]\\ \mathcal{O}}} \deg^+(D_{\mathcal{O}} - D') \\
&= \min_{\substack{D'\in[D]\\ \mathcal{O}}} (\deg(D) + \deg^+(D_{\mathcal{O}} - D') - (g-1)) - \min_{\substack{D'\in[D]\\ \mathcal{O}}} \deg^+(D_{\mathcal{O}} - D') \\
&= \deg(D) - (g-1).
\end{aligned}$$

□

If one hypothesizes a Riemann-Roch-like theorem for graphs, then this motivates the above definition of the genus. Namely, if we guess that

$$\mathrm{rk}(D) - \mathrm{rk}(K-D) = \deg(D) - \gamma + 1$$

for some γ depending only on G, then γ is easily found by letting $D = D_{\mathcal{O}}$ for some acyclic orientation $\mathcal{O}$. By Lemma 3.7, $\mathrm{rk}(D_{\mathcal{O}}) - \mathrm{rk}(D_{\bar{\mathcal{O}}}) = -1 - (-1) = 0$, so $\deg(D)_{\mathcal{O}} - \gamma + 1 = 0$ and $\gamma = \deg(D)_{\mathcal{O}} + 1 = |E| - |V| + 1$.

References

[Bak14] M. Baker. Reduced divisors and Riemann-Roch for graphs. Matt Baker's Math Blog, 2014. URL:https://mattbaker.blog/2014/01/12/reduced-divisors-and-riemann-roch-for-graphs/.

[BN07] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.