# TROPICAL BRILL-NOETHER THEORY

## 4. Rank determining sets

The material for today's talk has been adapted from Ye Luo's paper "Rank determining sets on metric graphs." (2009). We begin with some definitions.

**Definition 4.1.** *Let $D$ be a divisor on a graph $G$. We define the* complete linear series *of $D$ to be*

$$|D| := \{D' \sim D \mid D' \text{ is effective }\}.$$

In this language, a divisor $D$ has rank at least $r$ if, for every effective divisor $E$ of degree $r$, the complete linear series $|D - E|$ is nonempty.

**Definition 4.2.** *Let $D = \sum D(v)v$ be an effective divisor on a graph $G$. We define the* support *of $D$ to be*

$$\operatorname{supp}(D) := \{v \in V(G) \mid D(v) > 0\}.$$

*The support of the complete linear series $|D|$ is*

$$\operatorname{supp}(|D|) := \{v \in V(G) \mid D'(v) > 0 \text{ for some } D' \in |D|\}.$$

We say that a divisor $D$ has support in $A$ if $\operatorname{supp}(D)$ is contained in $A$.

Computing the rank of a divisor can be computationally intensive if one uses the naive method of checking whether $[D - E]$ is effective for all effective divisors $E$ of a given degree. This is especially true if the graph is sparse. Rank determining sets provide a more efficient way to calculate the rank, using only those effective divisors $E$ supported on a subset of the vertices.

**Definition 4.3.** *Let $G$ be a graph, and let $A$ be a subset of $V(G)$.*

*(1) The $A$-rank $r_A(D)$ is the largest integer $r$ such that $[D - E]$ is effective for all effective $E$ with support in $A$.*

*(2) The set $A$ is* rank determining *if $r_A(D) = r(D)$ for all $D \in \operatorname{Div}(G)$.*

**Remark 4.4.** Note that $r_A(D) \geq r(D)$ for any subset $A \subseteq V(G)$ and any divisor $D$.

**Definition 4.5.** *Let $A \subseteq V(G)$ be a set of vertices. We define $\mathcal{L}(A)$ to be the subset of $V(G)$ consisting of vertices $v$ with the property that, for any divisor $D$ with $A \subseteq \operatorname{supp}|D|$, we have $v \in \operatorname{supp}|D|$. In other words,*

$$\mathcal{L}(A) = \bigcap_{\operatorname{supp}|D| \supseteq A} \operatorname{supp}|D|.$$

**Proposition 4.6.** *Let $A$ be a nonempty subset of $V(G)$. The following are equivalent:*

*(1) $\mathcal{L}(A) = G$.*

*(2) If $D$ is a divisor with $r_A(D) \geq 1$, then $r(D) \geq 1$.*

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(3) *$A$ is a rank-determining set.*

*Proof.* We first show that (1) and (2) are equivalent. To see this, note that $\mathcal{L}(A) = G$ if and only if, for any divisor $D$ with $A \subseteq \text{supp}|D|$, we have $\text{supp}|D| = G$. Equivalently, for any such $D$, if $|D - v| \neq \emptyset$ for all $v \in A$, then $|D - v| \neq \emptyset$ for all $v \in V(G)$. But this is the same as saying that $r_A(D) \geq 1$ implies $r(D) \geq 1$.

The implication (3) implies (2) is clear from the definition of rank determining set.

It remains to show that (2) implies (3). Assume (2). It suffices to show that $r_A(D) \geq s$ implies $r(D) \geq s$ by Remark 4.4. We proceed by induction. The case $s = -1$ is trivial, since $r(D) \geq -1$ for all $D$. The case $s = 0$ is also clear, since $r_A(D)$ and $r(D)$ are both nonnegative if and only if $[D]$ is effective.

We proceed by induction on $s$. Suppose $s \geq 1$ and $r_A(D) \geq s$. Then

$$r_A(D - v) \geq s - 1$$

for all $v \in A$. From the induction hypothesis, we deduce that $r(D - v) \geq s - 1$. Therefore $[D - E - v]$ is effective for all effective $E$ of degree $s - 1$. Fixing $E$ and varying $v$, this means that $r_A(D - E) \geq 1$. Having assumed (2), we deduce that $r(D - E) \geq 1$. Allowing $E$ to vary over all effective divisors of degree $s - 1$, we conclude that $[D - E']$ is effective for all effective $E'$ of degree $s$, and hence $r(D) \geq s$, as required. □

Knowing that $A$ is a rank determining set if and only if $\mathcal{L}(A) = G$, we now provide a topological condition to determine when $\mathcal{L}(A) = G$. To do this we define a YL set.

**Definition 4.7.** *Let $G$ be a graph and $U \subseteq V(G)$ a subset of the vertices. We call $U$ a* YL set *if either $U = \emptyset$, $U = V(G)$, or the following conditions hold:*

(1) *The induced subgraph $G[U]$ is connected.*
(2) *Every connected component $X$ of the induced subgraph $G[U^c]$ contains a vertex $v$ such that* $\text{outdeg}_X(v) > 1$.

**Remark 4.8.** We have chosen to name YL sets after Ye Luo, whose work these notes are based upon. His paper is concerned with the case of metric graphs, which we will not see until a later lecture. Our choice of the letter $U$ to denote YL sets is motivated by the fact that their natural analogue in the setting of metric graphs is a certain type of connected open set.

We can characterize YL sets in terms of divisor theory.

**Lemma 4.9.** *Let $U \subseteq V(G)$ be a nonempty set of vertices and suppose that the induced subgraph $G[U]$ is connected. Let $\partial U$ be the set of neighbors of $U$ – that is, the set of vertices adjacent to a vertex in $U$, but not contained in $U$. Then $U$ is a YL set if and only if $D = \sum_{v \in \partial U} (v)$ is $w$-reduced for any $w \in U$.*

*Proof.* This follows immediately by applying Dhar's burning algorithm on any vertex $w \in U$. □

Given a divisor $D$ on $G$, we may use Lemma 4.9 to find YL sets that are disjoint from the support of $|D|$.

**Lemma 4.10.** *For $v \in V(G)$, let $D$ be a $v$-reduced divisor, and let $U$ be the set of vertices that can be reached from $v$ by a path that does not pass through* $\mathrm{supp}(D) \smallsetminus \{v\}$. *Then $U$ is a YL set. Moreover, if $v \notin \mathrm{supp}D$, then $U$ is disjoint from* $\mathrm{supp}|D|$.

*Proof.* Let $D' = \sum_{w \in \mathrm{supp}(D) \setminus v}(w)$. Since $D$ is $v$-reduced, we see that $D'$ is $v$-reduced as well. By Lemma 4.9, it follows that $U$ is a YL set. In addition, if $v \notin \mathrm{supp}(D)$, then by Dhar's burning algorithm we see that $D$ is $w$-reduced for all $w \in U$. It follows that $w \notin \mathrm{supp}|D|$ for all $w \in U$. □

The following consequence of Lemma 4.10 is not necessary for our other results on rank-determining sets, but may be of independent interest.

**Corollary 4.11.** *Let $D$ be a divisor on $G$. Then $(\mathrm{supp}|D|)^c$ is a disjoint union of YL sets.*

*Proof.* For each $v \in (\mathrm{supp}|D|)^c$, let $D_v$ be the $v$-reduced divisor equivalent to $D$, and let $U_v$ be the set of vertices that can be reached from $v$ by a path that does not pass through $\mathrm{supp}D_v \smallsetminus \{v\}$. By Lemma 4.10, $U_v$ is a YL set disjoint from $\mathrm{supp}|D|$. It follows that

$$(\mathrm{supp}|D|)^c = \bigcup_{v \in (\mathrm{supp}|D|)^c} U_v$$

is a union of YL sets.

To see that this union is disjoint, suppose that $U_v \cap U_{v'} \neq \emptyset$. Then, since $D_v$ is $w$-reduced for all $w \in U_v$ and $D_{v'}$ is $w$-reduced for all $w \in U_{v'}$, we see that $D_v = D_{v'}$ by uniqueness of reduced divisors. By the definition of $U_v$, we therefore see that $U_v = U_{v'}$. □

We now turn to the main theorem of this lecture, which gives a sufficient condition for subsets of the vertices to be rank-determining.

**Theorem 4.12.** *Let $v \in V(G)$ and $A$ be a nonempty subset of $V(G)$. Then $v \in \mathcal{L}(A)$ if, for all YL sets $U$ containing $v$, we have $A \cap U \neq \emptyset$. That is,*

$$\mathcal{L}(A) \supseteq \bigcap_{\substack{U \text{is } YL \\ A \cap U = \emptyset}} U^c.$$

*Moreover, $A$ is rank-determining if all nonempty YL sets intersect $A$.*

*Proof.* Let $D$ be a divisor such that $A \subseteq \mathrm{supp}|D|$. If $v \notin \mathrm{supp}|D|$, then by Lemma 4.10, there exists a YL set $U$ containing $v$ that is disjoint from $\mathrm{supp}|D|$. Thus, if all YL sets containing $v$ intersect $A$, then $A \subseteq \mathrm{supp}|D|$ implies that $v \in \mathrm{supp}|D|$. This gives us the containment above.

To see the final remark, note that $A$ is rank determining if and only if $\mathcal{L}(A) = G$. This happens if the only YL set appearing on the right hand side of the containment above is $\emptyset^c = V(G)$. In other words, the only YL set that does not intersect $A$ is the empty set. □

**Remark 4.13.** Luo proves the stronger result that the containment of Theorem 4.12 is in fact an equality, from which he derives that this sufficient condition for subsets to be rank-determining is also necessary. For our purposes, we will only need the fact that this condition is sufficient.

We note the following interesting property of YL sets.

**Lemma 4.14.** *If $G$ is a graph of genus $g$ and $U$ is a nontrivial YL set in $G$, then the induced subgraph $G[U \cup \partial U]$ has genus at least 1. As a consequence, there exist at most $g$ disjoint nonempty YL sets in $G$.*

*Proof.* If $G[U \cup \partial U]$ is a tree, then for every $v \in \partial U$, we have $\mathrm{indeg}_U(v) = 1$, and thus $U$ is not YL. $\square$

The condition for rank determining sets provided in Theorem 4.12 is useful for many reasons. We mention two consequences of this result below, which we will prove in a later lecture.

**Theorem 4.15.** *Let $G$ be a graph of genus $g$. Then there exists a rank-determining set of cardinality $g + 1$.*

**Remark 4.16.** For some graphs there may exist rank determining sets of smaller cardinality that $g + 1$. For example any three vertices form a rank determining set on the complete graph $K_4$, which has genus three.

We conclude with the following observation, which is an analogue for finite graphs of the topological invariance of the rank-determining property for metric graphs (to be discussed in a future lecture).

**Theorem 4.17.** *Rank determining sets are preserved under subdivision.*

In other words, if $A$ is a rank determining set on $G$, and $G'$ is obtained from $G$ by a series of subdivisions then $A$ is rank determining on $G'$. Here a subdivision of $G$ is obtained by specifying an edge $e$ of $G$ from $v$ to $w$ and constructing a new graph $G'$ with one new vertex $v'$, in which the edges of $G'$ are all of the edges of $G$ except for $e$, plus two new edges $[v, v']$ and $[v', w]$. Note that $V(G)$ is naturally identified with $V(G') \smallsetminus \{v'\}$, and that the geometric realization of $G'$ is naturally homeomorphic to that of $G$, by a morphism that preserves the vertices of $G$ and maps $v'$ to the midpoint of $e$.

*Proof.* Let $A \subset V(G)$ be a rank-determining set on $G$. We must show that $A$ is rank-determining on $G'$. By Theorem 4.12, every nonempty YL subset of $V(G)$ meets $A$, and it remains to prove that every nonempty YL subset of $V(G')$ meets $A$. We make an even stronger claim: If $U' \subset V(G')$ is a nonempty YL set on $G'$ then $U' \cap V(G)$ is a nonempty YL set on $G$. To see this, first note that the set $\{v'\}$ is not YL, so $U' \cap V(G)$ is nonempty. Now, let $X$ be a connected component of the induced graph $G[(U' \cap V(G))^c]$. Then either $X' = V(X)$ or $X' = V(X) \cup \{v'\}$ is the set of vertices of a connected component of $G'[U^c]$. Since the outdegree of $v'$ in this connected set is at most 1, we see that there exists an $x \in V(X)$ with $\mathrm{outdeg}_{X'}(x) > 1$. But $\mathrm{outdeg}_X(x) \geq \mathrm{outdeg}_{X'}(x)$ for all $x \in V(G)$, so the claim follows. $\square$

## References

[Luo11] Y. Luo. Rank-determining sets of metric graphs. *J. Combin. Theory Ser. A*, 118:1775–1793, 2011.