TROPICAL BRILL-NOETHER THEORY

5. Special divisors on a chain of loops

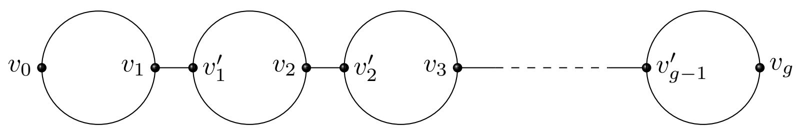
For this lecture, we will study special divisors a generic chain of loops. More specifically, when g, r, d are nonnegative numbers satisfying

$$\rho(g, r, d) = g - (r + 1)(d - g + r) = 0,$$

then we give a complete classification of the divisor classes of degree d and rank r on a generic chain of g loops.

Reference: F. Cools, J Draisma, S. Payne, E. Robeva: "A tropical proof of the Brill-Noether Theorem" Adv. in Math. 230 (2012) 759-776.

Let G be a chain of g loops with bridges:



with m_i edges from v'_{i-1} to v_i counterclockwise, ℓ_i edges v_i to v'_{i-1} counterclockwise. That is, we can express G as

$$G = \gamma_1 \cup \beta_1 \cup \gamma_2 \cup \cdots \cup \beta_{g-1} \cup \gamma_g$$

where γ_i denotes the ith loop and β_i is the bridge connecting γ_i to γ_{i+1}. The bottom part of γ_i consists of m_i edges and the top part of γ_i consists of ℓ_i edges. Here is the main theorem in this section:

Theorem 5.1. *Suppose G is a generic chain of loops, and let g, r, d be nonnegative integers such that $\rho(g, r, d) := g - (r + 1)(g - d - r) = 0$. Then there are exactly*

$$\lambda(g, r, d) = g! \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}$$

divisor classes of degree d and rank r on G.

We will prove this theorem by constructing a bijection between divisor classes of degree d and rank r, and standard tableaux on the $(r + 1) \times (g - d + r)$ rectangle (note that λ is the number of standard tableaux on this box, by the hook length formula). Before discussing the proof of this theorem, let's characterize reduced divisors and rank determining sets on the chain of loops.

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Proposition 5.2. *(Characterization of v_n-reduced divisors on the chain of loops): A divisor D on G is v_n-reduced if and only if it is effective away from v_n, has no chips in the interiors of bridges, and there is at most one chip on each $\gamma_i \smallsetminus v_i$ for $i \leq n$ and $\gamma_i \smallsetminus v'_{i-1}$ for $i > n$*

Proof. Apply Dhar's algorithm to v_n. □

Remark 5.3. To give a v_0-reduced divisor on G, it is enough to specify the number of chips placed at v_0 and the location of the unique chip of on $\gamma_i \smallsetminus v'_{i-1}$ if one exists. We specify the location of a chip on $\gamma_i \smallsetminus v'_{i-1}$ by its distance away from v'_{i-1} in the counterclockwise direction measured in the number of vertices traversed (so the distance from v'_{i-1} to v_i is m_i). Therefore, there is a one-to-one correspondence between v_0-reduced divisors on G and $\mathbb{Z}\times\mathbb{Z}/(\ell_1+m_1)\times\cdots\times\mathbb{Z}/(\ell_g+m_g)$; given a v_0 reduced divisor D, associate $(d_0; a_1, \ldots, a_g)$ where $D(v_0) = d_0$ and a_i is the distance of the unique chip of D on γ_i, if there is one, and zero otherwise.

Proposition 5.4. *(Rank determining sets for G) A divisor D on G has rank at least r if and only if $[D-E]$ is effective for every effective divisor $E = r_0v_0+\cdots+r_gv_g \geq 0$ of degree r. In other words, $\{v_0, \ldots, v_g\}$ is a rank determining set.*

Proof. Let Γ be the metric graph associated to G. Then the closures of the connected components of $\Gamma \smallsetminus \{v_0, \ldots, v_g, v'_1, \ldots, v'_{g-1}\}$ are contractible, so by a theorem of Ye Luo, $\{v_0, \ldots, v_g, v'_1, \ldots, v'_{g-1}\}$ is a rank determining set. Moreover, $v'_{i-1} \sim v_i$ for $1 \leq i \leq g-1$, so $\{v_0, \ldots, v_g\}$ is a rank determining set. □

Proposition 5.5. *(Divisors on one loop) Suppose g=1, so G is the loop on $\ell+m$ vertices (vertices $w_0, \ldots, w_m, w_{1+m}, \ldots, w_{\ell+m} = w_0$ going around counterclockwise). Let $D = kw_0 + u$, where $k \geq 0$, $\ell+m$ does not divide km and u the zero divisor or $u = w_i$ $(i \neq 0)$. Then $D \sim D'$ where*

$$D' = \begin{cases} (k-1)w_m + w_{-(k-1)m} & u = 0 \\ (k+1)w_m & u = w_{(k+1)m} \\ kw_m + w_{i-km} & \textit{otherwise} \end{cases}.$$

Proof. Apply Dhar's burning algorithm to w_m. □

Example 5.6. Let $g \geq 2$ and suppose G is a loop on $2g-1$ vertices (this is the case $\ell = 2g-2$ and $m = 1$ in the above proposition). Let $D = kw_0 + u$ where $1 \leq k \leq 2g-3$ and u is either the zero divisor or $u = w_i$ for $i \neq 0$. Then $D \sim D'$ where

$$D' = \begin{cases} (k-1)w_1 + w_{-(k-1)} & u = 0 \\ (k+1)w_1 & u = w_{k+1} \\ kw_1 + w_{i-k} & \text{otherwise} \end{cases}.$$

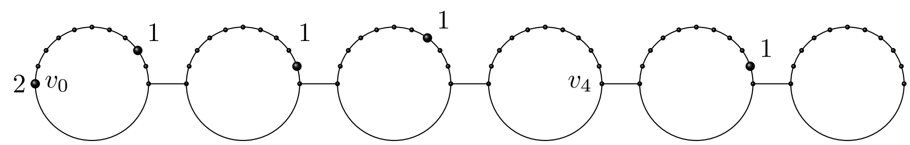
Now let us assume that $g \geq 2$. Let r and d be such that $\rho(g, r, d) = 0$ and assume that $d \leq 2g-2$ (d here will be the degree of our divisors, and Riemann-Roch forces the rank to be $d-g$ when $d \geq 2g-1$) Furthermore, we assume that the chain of loops is generic in the following sense:

Definition 5.7. *The chain of loops G is generic if ℓ_i/m_i cannot be expressed as a fraction of integers whose absolute value sum to at most $2g-2$.*

For example, we may take $\ell_i = 2g-2$ and $m_i = 1$, for all i.

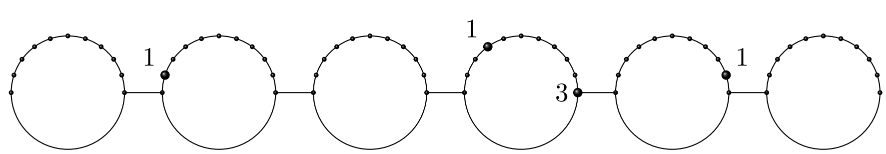
We are now ready to classify divisor classes on G of degree d and rank r. First, let's think of this classification problem in the context of the following game played by Baker and Norine. Start with r chips on v_0. Baker places $d-r$ chips on the loops $\gamma_i \smallsetminus v_{(i-1)}$, at most 1 chip per loop. Norine places r anti-chips among the vertices $v_0, \ldots, v_g$. Baker wins if he can move the pile of chips from v_0 across the graph (according to the rules in Proposition 5.5) to annihilate each of Norine's anti-chips. Finding all divisor classes of degree d and rank r amounts to classifying Baker's winning strategies. Here is an example of a winning strategy for $(g,r,d) = (6,2,6)$.

Example 5.8. Let $(g,r,d) = (6,2,6)$ (observe that $\rho(g,r,d) = 0$). Let G be the chain of 6 loops where each loop has 11 edges: $m=1$ edge on the bottom and $\ell = 10$ edges on the top. Then the following divisor D has degree 6 and rank 2:



We must check that $r(D) = 2$. By Proposition 5.2 D is v_0-reduced. Since $D(v_0) = 2$, we have that $r(D) \leq 2$. To show that $r(D) \geq 2$, we can use Proposition 5.4. We must check that $[D-E]$ is effective for any effective divisor E of degree 2 whose support is contained in $\{v_0, v_1, \ldots, v_6\}$. To check that each $[D-E]$ is effective, start with the pile of chips at v_0 and move them across each loop according to the rules of Proposition 5.5. For example, suppose $E = -2 \cdot v_4$. Starting with the divisor D, we wish to move the pile of chips at v_0 over to v_4 so that we are left with a divisor D' in $[D]$ with $D'(v_4) \geq 2$. This will give us an effective divisor $D' - E$ in $[D-E]$.

First, let us move the pile from v_0 to v_1. We are in the second case of Proposition 5.5, so we will have 3 chips at v_1'. Since $v_1' \sim v_1$, we can move these 3 chips across the bridge β_1. Now we must move the pile of 3 chips from v_1 to v_2'. Here, we are in the third case of Proposition 5.5, we can move 3 chips to v_2'. Again, we can move this pile of chips across β_2 to have 3 chips at v_2. Moving the pile of chips from v_2 to v_3' we are again in the second case of Proposition 5.5, so the pile of chips grows to 4 chips at v_3'. Moving this pile across β_3, we are left with traversing the 4th loop. We are in the first case of Proposition 5.5, so we can only move 3 chips to v_4. This divisor is our D' as shown below.



After subtracting E from this divisor, we are left with an effective divisor in $[D-E]$.

Note that we most perform this check for each one of the $\binom{7+1}{2} = 28$ eligible divisors E. This approach is cumbersome, so instead we will develop a way to keep track of how the pile of chips grows and shrinks as it moves across the chain of loops. This information will be recorded in the lingering lattice path of a divisor on the chain of loops, which we will now define.

Definition 5.9. *A lingering lattice path in $\mathbb{Z}^r$ is a sequence $P_0, \dots, P_g$ of points in $\mathbb{Z}^r$ such that $P_i - P_{i-1}$ is either a standard basis vector, the zero vector, or $(-1, \dots, -1)$.*

In $\mathbb{Z}^r$, label the coordinates $0, \dots, r-1$ and let $P_i(j)$ be the jth coordinate of P_i. We write $\mathcal{C}$ for the chamber

$$\mathcal{C} = \{y \in \mathbb{Z}^r | y(0) > y(1) > \cdots > y(r-1) > 0\}.$$

Thus elements of $\mathcal{C}$ are points whose coordinates are strictly decreasing positive integers. Each v_0-reduced divisor is assigned a lingering lattice path by the following rules.

Definition 5.10. *Let D be a v_0-reduced divisor of degree d_0 on G, and $(d_0; a_1 \dots, a_g)$ the coordinates associated to D as in Remark 5.3. Then the lingering lattice path associated to D is the sequence $P_0, \dots, P_g$ where $P_0 = (d_0, d_0 - 1, \dots, d_0 - r + 1)$ and*

$$P_i - P_{i-1} = \begin{cases} (-1, \dots, -1) & a_i = 0 \\ e_j & a_i \equiv (P_{i-1}(j) + 1)m_i \mod m_i + \ell_i \\ & \text{and } P_{i-1}, P_{i-1} + e_j \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}.$$

Let's check that P_i is well defined. Suppose $P_0, \dots, P_{i-1}$ are well defined. The coordinates of P_{i-1} are a strictly decreasing sequence of integers. This is because the coordinates of P_0 are strictly decreasing and each step preserves these strict inequalities. Let b_{i-1} be the number of steps in the $(-1, \dots, -1)$ direction in the path $\{P_0, \dots, P_{i-1}\}$. Since there are only $d - d_0$ loops that have a chip, there can be at most $d - d_0$ steps in the e_0 direction, so

$$P_{i-1}(0) \le P_0(0) + d - d_0 - b_{i-1} \le 2g - 2 - b_{i-1}.$$

On the other end, $P_0(r-1) \ge 1$, so $P_{i-1}(r-1) \ge 1 - b_{i-1}$. Altogether, this means that the coordinates of P_{i-1} form a strictly decreasing sequence of r distinct integers among the $2g-2$ integers in $\{1 - b_{i-1}, \dots, 2g - 2 - b_{i-1}\}$. The genericity condition ensures that

$$a_i \equiv (P_{i-1}(j) + 1)m_i \mod m_i + \ell_i$$

holds for at most one j, so the ith step is well defined.

Example 5.11. For the divisor in Example 5.8, the associated lingering lattice path is $P_0 = (2,1)$, $P_1 = (3,1)$, $P_2 = (3,2)$, $P_3 = (4,2)$, $P_4 = (3,1)$, $P_5 = (3,2)$, $P_6 = (2,1)$.

Theorem 5.12. *A v_0-reduced divisor D has rank at least r if and only if its associated lingering lattice path in $\mathbb{Z}^r$ lies entirely in*

$$\mathcal{C} = \{y \in \mathbb{Z}^r | y(0) > y(1) > \cdots > y(r-1) > 0\}.$$

Let's use this theorem to prove Theorem 5.1.

Proof of Theorem 5.1. First let's construct a bijection

$$\left\{ \begin{array}{c} \text{divisor classes of} \\ \text{degree } d \text{ and rank } r \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} g \text{ step lattice paths from} \\ (r,\ldots,1) \text{ to itself in } \mathcal{C} \end{array} \right\}.$$

Given a v_0-reduced divisor of degree d and rank r, assign its associated lingering lattice path $P_0, \ldots P_g$. This path lies entirely in $\mathcal{C}$ by Theorem 5.12. Since we start at $P_0 = (r, \ldots, 1)$ the only thing we need to show is that $P_g = (r, \ldots, 1)$. This follows from the fact that there are exactly $g - d + r$ steps in each coordinate direction and in the $(-1, \ldots, -1)$ direction. We can see this as follows. There are at least $g-d+r$ loops $\gamma_i \smallsetminus v'_{i-1}$ that have no chip, so there are $g - d + r$ steps in the $(-1, \ldots, -1)$ direction. For the remaining directions, first note that $P_0(r-1) = 1$ implies that

$$P_g(r-1) = 1 - (g - d + r) + \#\{e_{r-1} \text{ steps}\}$$

By Theorem 5.12, this quantity must be positive, so there must be at least $g - d + r$ steps in the e_{r-1} direction. Since the number of steps in the remaining coordinate directions is at least the number of steps in the e_{r-1} direction, we have

$$g = \text{ total number of steps } \geq (r+1)(g-d+r) = g,$$

so there must be exactly $g - d + r$ steps in each coordinate direction.

On the other hand, given a g-step lattice path $P_0, \ldots, P_g$ of $(r, \ldots, 1)$ to itself in $\mathcal{C}$, associate to it the following v_0-reduced divisor D. First, place r chips on v_0. If the ith step is in the e_j coordinate direction, place a chip on the ith loop precisely $(P_{i-1}(j) + 1)m_i$ vertices counterclockwise from v'_{i-1}. This divisor has degree d since there are r chips at v_0 and $d - r$ steps are in a coordinate direction. The lattice path associated to D is precisely $P_0, \ldots, P_g$, so, by Theorem 5.12, the divisor D has rank at least r. Since D is v_0-reduced and $D(v_0) = r$, D must have rank equal to r.

Now let's construct a bijection between lattice paths and tableaux

$$\left\{ \begin{array}{c} g \text{ step lattice paths from} \\ (r,\ldots 1) \text{ to itself in } \mathcal{C} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{standard tableaux on the Young} \\ \text{diagram with } r+1 \text{ columns and} \\ g-d+r \text{ rows} \end{array} \right\}$$

Label the columns of our Young diagram 0 to r from left to right. Given a g-step lattice path $P_0, \ldots, P_g$ from $(r, r-1, \ldots, 1)$ to itself in $\mathcal{C}$, construct the following tableau T. If the ith step is in the jth coordinate direction, put the number i in the jth column (filling the Young diagram from top to bottom). If the ith step is in the $(-1 \ldots, -1)$ direction, then put the number i in the last column. We claim that this T is a standard tableau. The numbers are strictly increasing in the columns by design. After i steps, the number of e_j steps must be at most the number of $e_{j'}$ steps for $j' < j$ (as our lattice path lies entirely in $\mathcal{C}$). Similarly the number of e_{r-1} steps must be at least the number of $(-1, \ldots, -1)$-steps. This shows that the rows of T are strictly increasing, as required.

On the other hand, given a tableau T, construct the following lattice path. Start with $P_0 = (r, \ldots, 1)$. If i is in the jth column for $j \leq r - 1$, then set $P_i = P_{i-1} + e_j$. If i is in the last column, then set $P_i = P_{i-1} + (-1, \ldots, -1)$. This is a lattice path from $(r, \ldots, 1)$ to itself that lies entirely in $\mathcal{C}$. ∎

Let's consider some examples illustrating these bijections.

Example 5.13. The divisor in Example 5.8 corresponds to the tableau

1	2	4
3	5	6

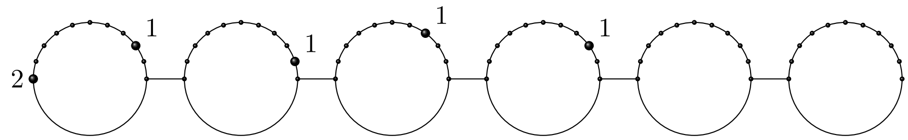
.

Example 5.14. Let G, and (g, r, d) be as in the previous example, and T the tableau

1	2	5
3	4	6

.

The corresponding lingering lattice path is $P_0 = (2, 1)$, $P_1 = (3, 1)$, $P_2 = (3, 2)$, $P_3 = (4, 2)$, $P_4 = (4, 3)$, $P_5 = (3, 2)$, $P_6 = (2, 1)$. To this we associate the following divisor on G.



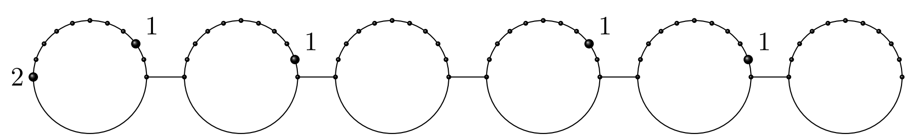
Given a tableau T we can produce a recipe for constructing a degree d and rank r divisor that is v_0 reduced. Start with r chips at v_0. Suppose we have placed chips appropriately on the loops $\gamma_1 \setminus v_1, \ldots, \gamma_{i-1} \setminus v_{i-1}$. If i is in the jth column, then subtract j chips from v_0. Then a chip is placed at the unique location on $\gamma_i \setminus v'_{i-1}$ so that the pile of chips has its maximum size when it reaches v_i. When i is in the rth column, subtracting r chips from v_0 will result in a v_i-reduced divisor. In this case, no chip is placed on $\gamma_i \setminus v_i$. Let's illustrate this in the following example.

Example 5.15. Let G and (g, r, d) be the same as in the previous examples, and T the tableau

1	2	3
4	5	6

.

Let's construct the divisor associate to this tableau. Start with $r = 2$ chips at v_0. Since 1 is in the zeroth column, we subtract zero chips from v_0. To get a pile of maximum size at v_1 (in this case, we are looking to get 3 chips at v_1), place a chip two vertices counterclockwise from v_1. Now let's consider the chip placement on the second loop. Since 2 is in the first column of T, we subtract one chip from v_0. According to the rules of Proposition 5.5, we can only move one chip from v_0 to v_1. We must place a chip one vertex counterclockwise from v_2 to be able to achieve 2 chips to v_2. Now let's consider the third loop. Since 3 is in the second column of T, do not put any chip on $\gamma_3 \setminus v_3$. Continuing in this way, we will get the divisor pictured below.



The sizes that the pile can achieve in the steps of this recipe are recorded in the lingering lattice path. More precisely, $P_n(j)$ is the smallest the pile can get at v_n when we play the subtract j chips game. These remarks are formalized in the following lemma which will be used to prove Theorem 5.12.

Lemma 5.16. *Let D be a divisor of degree d_0 on G and $P_0, \ldots, P_{n-1} \in \mathcal{C}$. Let $E_n \geq 0$ be an effective divisor of degree $j < r$, supported in $\{v_0, \ldots, v_n\}$ and D_n the v_n-reduced representative in $[D - E_n]$. Then*
(1) $D_n(v_n) \geq P_n(j)$.
(2) $D_n|_{\gamma_i \smallsetminus v'_{i-1}} = D|_{\gamma_i \smallsetminus v'_{i-1}}$ for $i > n$.

Furthermore, there exists E_n of degree j with support in $\{v_0, \ldots, v_n\}$ such that we get equality in (1).

Proof. We prove this by induction on n. Let $E_n = r_0 v_0 + \cdots + r_n v_n$ be an effective divisor of degree $j < r$. For $n = 0$, $D_0 = D - r_0 v_0$ is already v_0-reduced. $D_0(v_0) = d_0 - r_0 = P_0(r_0)$. (2) is also clear.

Now suppose that $D_{n-1}(v_{n-1}) \geq P_{n-1}(j - r_n)$ and D_{n-1} agrees with D on the loops $\gamma_i \smallsetminus v'_{i-1}$ to the right of γ_n. Let $D'_{n-1} \sim D_{n-1}$ that is v_n-reduced and set $D_n = D'_{n-1} - r_n v_n$. This is still v_n-reduced and property (2) is clear. We must show that D_n satisfies (1). We will do this by considering cases parallel to the options in Proposition 5.5.

Case 1: Suppose D has no chip on $\gamma_n \smallsetminus v'_{n-1}$. Then $P_n = P_{n-1} + (-1, \ldots, -1)$ and we are in the first case in Proposition 5.5, so

$$D_n(v_n) = D_{n-1}(v_{n-1}) - r_n - 1 \geq P_{n-1}(j - r_n) - r_n - 1 \geq P_{n-1}(j) - 1 = P_n(j).$$

Case 2: Suppose D has a chip that is $(P_{n-1}(j - r_n) + 1)m_n$ vertices counterclockwise from v'_{n-1}. If $D_{n-1}(v_{n-1}) = P_{n-1}(j - r_n)$, then we are in the second case of Proposition 5.5 so we pick up a chip.

$$D_n(v_n) = D_{n-1}(v_{n-1}) - r_n + 1 = P_{n-1}(j - r_n) - r_n + 1 \geq P_{n-1}(j) + 1 \geq P_n(j).$$

If $D_{n-1}(v_{n-1}) > P_{n-1}(j - r_n)$, then we are in the third case of Proposition 5.5, so the pile doesn't change size as we move it across the nth loop.

$$D_n(v_n) = D_{n-1}(v_{n-1}) - r_n \geq P_{n-1}(j - r_n) - r_n + 1 \geq P_{n-1}(j) + 1 \geq P_n(j).$$

Case 3: Suppose D has a chip on γ_n that is not $(P_{n-1}(j - r_n) + 1)m_n$ vertices counterclockwise from v'_{n-1}. In this case the pile cannot shrink as it moves across the nth loop and $P_n = P_{n-1}$. So,

$$D_n(v_n) \geq D_{n-1}(v_{n-1}) - r_n \geq P_{n-1}(j - r_n) - r_n \geq P_{n-1}(j) = P_n(j).$$

It remains to show that there is an E_n so that we have equality in (1). When $n = 0$, $E_n = j v_0$ works. For the inductive step, suppose r_n is the largest number so that the entries of P_n at positions $j - r_n, \ldots, j$ are consecutive integers. Let E_{n-1} be an effective divisor of degree $j - r_n$ with support in $\{v_0 \ldots, v_{n-1}\}$ so that the coefficient of v_{n-1} of the v_{n-1}-reduced divisor in $[D - E_{n-1}]$ equals $P_{n-1}(j - r_n)$. Then $E_n := E_{n-1} + r_n v_n$ is the required divisor. This can be seem by running through the inequalities in the cases above. □

Proof of Theorem 5.12. Suppose that the lattice path $P_0, \ldots P_g$ lies in $\mathcal{C}$. Let $E = r_0 + \cdots + r_g v_g$ be an effective divisor of degree r, and n the largest index so that $r_n > 0$. Let $E_n = E - v_n$. By the previous lemma, $D - E_n$ is equivalent to an effective divisor D'_n such that $D'_n(v_n) \geq P_n(r - 1)$. So $D - E$ is equivalent to the effective divisor $D'_n - v_n$. This shows that $r(D) \geq r$ in light of Proposition 5.4.

Conversely, suppose that the lattice path $P_0, \dots P_g$ does not lie in $\mathcal{C}$. Let n be the smallest index such that $P_n \notin \mathcal{C}$. By the construction of this path, this means that all coordinates of P_n are positive except for the last one where $P_n(r-1) = 0$. By the lemma, there is an effective divisor $E_n = r_0 v_0 + \cdots + r_n v_n$ of degree $r-1$ such that $D'_n(v_n) = 0$ where D'_n is the v_n-reduced divisor in $[D - E_n]$ as before. Then $E = E_n + v_n$ is an effective divisor of degree r such that $[D - E]$ is not effective (which can be seen by considering the v_n-reduced divisor in this class). □

References

[CDPR12] F. Cools, J. Draisma, S. Payne, and E. Robeva. A tropical proof of the Brill-Noether theorem. *Adv. Math.*, 230(2):759–776, 2012.