TROPICAL BRILL-NOETHER THEORY

7. Specialization of Divisors from Curves to Graphs

We discuss a tool for connecting divisor theory on algebraic curves to divisor theory on finite graphs. References include the paper "Specialization of linear systems from curves to graphs" by Baker and the textbook "Algebraic Geometry and Arithmetic Curves" by Qing Liu.

We begin by fixing some notation. Throughout, R will denote a complete discrete valuation ring with field of fractions K and algebraically closed residue field k. Let X be a smooth, proper, geometrically connected curve over K.

Definition 7.1. *A curve $\mathfrak{X}$ over R is called a* strongly semistable model *for X over R if it satisfies the following:*

(1) $\mathfrak{X}$ is proper and flat over R.
(2) $\mathfrak{X}$ is regular.
(3) The generic fiber $\mathfrak{X}_K$ is X.
(4) The special fiber $\mathfrak{X}_k$ is reduced, has smooth components and only ordinary double points as singularities.

We think of $\mathfrak{X}$ as an infinitesimal family of smooth curves X degenerating to a singular curve $\mathfrak{X}_k$. We write $\mathcal{C} = \{C_1, \ldots, C_n\}$ be the set of irreducible components of $\mathfrak{X}_k$.

Remark 7.2. We make the following observations:

(1) Because X is smooth and $\mathfrak{X}$ is regular, all Weil divisors on X or $\mathfrak{X}$ are Cartier.
(2) Because $\mathfrak{X}$ is flat over R, for any line bundle $\mathcal{L}$ on $\mathfrak{X}$, we have

$$\deg \mathcal{L}|_X = \deg \mathcal{L}|_{\mathfrak{X}_k},$$

and because $\mathfrak{X}_k$ is reduced,

$$\deg \mathcal{L}|_{\mathfrak{X}_k} = \sum_{i=1}^{n} \deg \mathcal{L}|_{C_i}.$$

(3) Because $\mathfrak{X}$ is flat over R, we have that X is dense in $\mathfrak{X}$, hence there is an isomorphism of function fields

$$K(X) \cong K(\mathfrak{X}).$$

Definition 7.3. *The* dual graph *G of $\mathfrak{X}_k$ is the graph defined as follows:*

(1) The vertices $V(G) = \{v_1, \ldots, v_n\}$ correspond to the components C_i of $\mathfrak{X}_k$.
(2) For $i \neq j$, the number of edges between v_i and v_j is the number of points in $C_i \cap C_j$.

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Note that $\mathcal{C} \subset \mathrm{Div}(\mathfrak{X})$ and there is an intersection product

$$\mathcal{C} \times \mathrm{Div}(\mathfrak{X}) \to \mathbb{Z}$$

given by

$$C_i \cdot \mathcal{D} = \deg(\mathcal{O}_{\mathfrak{X}}(\mathcal{D})|_{C_i}).$$

Because $\mathfrak{X}$ is semistable, the number of edges between v_i and v_j in G is equal to $C_i \cdot C_j$.

Definition 7.4. *We define the* specialization map *from $\mathfrak{X}$ to G as*

$$\rho : \mathrm{Div}(\mathfrak{X}) \to \mathrm{Div}(G)$$

$$\rho(D) = \sum_{i=1}^{n} (C_i \cdot \mathcal{D}) v_i.$$

The specialization map has the following nice properties.

Proposition 7.5.

(1) If $\mathcal{D} \in \mathrm{Prin}(\mathfrak{X})$, then $\rho(\mathcal{D}) = 0$.
(2) If $\mathcal{F} \in \mathrm{Div}(\mathfrak{X})$ is vertical *(meaning supported in $\mathfrak{X}_k$) then $\rho(\mathcal{F}) \in \mathrm{Prin}(G)$.*

Proof.

(1) This is clear because, if $\mathcal{D} \in \mathrm{Prin}(\mathfrak{X})$, then $\mathcal{O}(\mathcal{D}) = \mathcal{O}_{\mathfrak{X}}$.
(2) It suffices to show that $\rho(C_j)$ is principal for all $C_j \in \mathcal{C}$. Write

$$\rho(C_j) = \sum_i a_i v_i.$$

Then for $i \neq j$, $a_i = C_i \cdot C_j$ is the number of edges between v_i and v_j. Also because C_j is supported in $\mathfrak{X}_k$,

$$\begin{aligned} 0 &= \deg \mathcal{O}(C_j)|_X \\ &= \deg \mathcal{O}(C_j)|_{\mathfrak{X}_k} \\ &= \sum_{i=1}^{n} \deg \mathcal{O}(C_j)|_{C_i} \\ &= \deg \rho(C_j). \end{aligned}$$

Therefore solving for a_j, we see that

$$a_j = -\operatorname{val}(v),$$

so $\rho(C_j)$ is the principal divisor obtained by firing vertex v_j.

□

We have a map $\mathrm{cl} : \mathrm{Div}(X) \to \mathrm{Div}(\mathfrak{X})$ given by taking closure of prime divisors and extending linearly. We define the *specialization map* from X to G as the composition

$$\rho \circ \mathrm{cl} : \mathrm{Div}(X) \to \mathrm{Div}(G).$$

By abuse of notation, we also call this specialization map ρ. This specialization map has the following nice properties.

Proposition 7.6.

(1) If $E \in \mathrm{Div}(X)$ is effective, then $\rho(E)$ is effective.
(2) If $D \in \mathrm{Div}(X)$, then

$$\deg(D) = \deg(\rho(D)).$$

(3) If $D \in \mathrm{Div}(X)$ is principal, then $\rho(D)$ is principal.

Proof.

(1) This is clear because the intersection product of two distinct prime divisors is nonnegative.
(2) We have

$$\deg(D) = \deg(\mathcal{O}(\mathrm{cl}\, D)|_X) = \deg(\mathcal{O}(\mathrm{cl}\, D)|_{\mathfrak{X}_k}) = \deg(\rho(D)).$$

(3) If $D = \mathrm{div}(f)$ for some $f \in K(X) = K(\mathfrak{X})$, then in $\mathfrak{X}$,

$$\mathrm{cl}\, D = \mathrm{div}\, f + \mathcal{F},$$

for some vertical divisor $\mathcal{F}$. Thus by Proposition 7.5, it follows that $\rho(D)$ is principal.

□

We also have the following surjectivity result for specialization, which we do not prove here. Details can be found in Liu's "Algebraic Geometry and Arithmetic Curves".

Proposition 7.7. *Let $Q \in \mathfrak{X}_k$ be a smooth point of the special fiber. Then there exists $P \in X(K)$ such that P specializes to Q in $\mathfrak{X}$.*

Remark 7.8. Proposition 7.7, which is essentially Hensel's Lemma, requires our assumptions that $\mathfrak{X}$ is regular and that the residue field k is algebraically closed.

This result gives the following corollary that will be critical in the proof of the specialization lemma.

Corollary 7.9.

(1) For all $v \in V(G)$, there exists $P \in X(K)$ such that $\rho(P) = v$.
(2) X has infinitely many K-points.

The specialization lemma will tell us how rank behaves under specialization. To prove it, we will first give an alternative definition for the rank of a divisor on a curve. We will then show that in our setting, this definition is equivalent to the usual notion of rank.

Definition 7.10. *Let $\mathrm{Div}(X(K)) \subset \mathrm{Div}(X)$ be the free abelian group on the K-points of X. For $D \in \mathrm{Div}(X)$, we define $r_K(D)$ to be the largest integer r such that $|D - E| \neq \emptyset$ for all effective divisors E of degree r.*

Proposition 7.11. *For $D \in \mathrm{Div}(X(K))$, $r_K(D) = h^0(X, \mathcal{O}(D)) - 1$, the dimension of $|D|$.*

Proof. Let $h^0(X, \mathcal{O}(D)) = r + 1$. We have that for any $P \in X(K)$,

$$h^0(X, \mathcal{O}(D - P)) \geq h^0(X, \mathcal{O}(D)) - 1.$$

By induction, for any effective divisor E of degree r, we have

$$h^0(X, \mathcal{O}(D-E)) \geq 1.$$

Thus $|D-E| \neq \emptyset$, so $r_K(D) \geq r$.

For the reverse inequality, we induct on r. When $r=-1$, we have that D is not equivalent to any effective divisors, so $r_K(D)=-1$. Now suppose that $r \geq 0$. Define

$$\mathcal{L}(D) = \{f \in K(X) | \operatorname{div}(f) + D \geq 0\} \cup \{0\} \cong H^0(X, \mathcal{O}(D)).$$

Because $\dim \mathcal{L}(D) = h^0(X, \mathcal{O}(D)) \geq 1$, there exists a nonzero $f \in \mathcal{L}(D)$. Because $\operatorname{supp}(D)$ is finite, the number of zeros of f is finite, and $X(K)$ is infinite, there exists $P \in X(K) \smallsetminus \operatorname{supp}(D)$ such that $f(P) \neq 0$. Thus $f \notin \mathcal{L}(D-P)$, so

$$h^0(X, \mathcal{O}(D-P)) < h^0(X, \mathcal{O}(D)) = r+1.$$

By induction $r(D-P) < r$, so there exists an effective divisor E of degree r such that

$$|D-P-E| = \emptyset.$$

Therefore $r(D) < r+1$. □

We can now state and prove the specialization lemma.

Lemma 7.12 (Specialization Lemma)**.** *For any divisor D in X*

$$r(\rho(D)) \geq r(D).$$

Proof. Let $r = r(D)$ and $\overline{E} \in \operatorname{Div}_+^r(G)$. By Corollary 7.9, there exists $E \in \operatorname{Div}_+^r(X(K))$ such that $\overline{E} = \rho(E)$. Because $r(D) = r$, we have that $D - E$ is linearly equivalent to an effective divisor. Then by Proposition 7.6, we have that $\rho(D) - \overline{E}$ is equivalent to an effective divisor. Thus $r(\rho(D)) \geq r(D)$. □

References

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