TROPICAL BRILL-NOETHER THEORY

9. Jacobians of metric graphs

In algebraic geometry, given a smooth algebraic curve C of genus g, one associates an abelian variety known as the *Jacobian* of C denoted $\mathrm{Jac}(C)$. Over the complex numbers, $\mathrm{Jac}(C)$ is simply a complex structure on the real manifold $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. In the piecewise linear setting, we work instead with metric graphs. As we will see in Lecture 11, and might guess from Lecture 7, these graphs arise from one-parameter families of algebraic curves. The associated family of Jacobians degenerates when the curve degenerates, and the combinatorics of its degeneration is captured by the tropical Jacobian, which we discuss today. The primary references for this section are the papers of Baker and Norine on Abel-Jacobi theory for finite graphs, the paper on Mikhalkin and Zharkov establishing the metric theory, and the paper of Baker and Faber on the metric properties of the tropical Abel-Jacobi map.

Throughout this lecture, we will use Γ to refer to a metric graph of genus g. Let G be a model of Γ with the associated length function $\ell : E(G) \to \mathbb{R}_{>0}$. We define the *Jacobian* of Γ as a quotient of singular homology groups,

$$J(\Gamma) := H_1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z}).$$

It is a real torus of dimension g.

9.1. **CW-Homology and Harmonic 1-forms.** The torus $J(\Gamma)$ is canonically identified with the quotient of CW-homology groups $H_1(G, \mathbb{R})/H_1(G, \mathbb{Z})$. Indeed, by the universal coefficients theorem, the CW-homology of G is equivalent to the singular homology of Γ, and gives more direct access to the geometry of $J(\Gamma)$. We review this now.

Fix an orientation on each edge of G. The 0-*chains* of G are the elements of $C_0(G, \mathbb{R}) := \mathbb{R}^{V(G)}$, where $V(G)$ is the vertex set of G. Similarly, the 1-*chains* of G are the elements of $C_1(G, \mathbb{R}) := \mathbb{R}^{E(G)}$. A 0-chain (resp. 1-chain) has *integral coefficients* if it is contained in $\mathbb{Z}^{V(G)}$ (resp. $\mathbb{Z}^{E(G)}$). We let $C_0(G, \mathbb{Z})$ (resp. $C_1(G, \mathbb{Z})$) denote the subset of 0-chains (resp. 1-chains) with integral coefficients.

Given an edge e, we use v_e^- to denote its *initial vertex*, and v_e^+ to denote its *terminal vertex* in the chosen orientation. The *boundary map* $\partial : C_1(G, \mathbb{R}) \to C_0(G, \mathbb{R})$ is determined by setting $\partial(e) = v_e^+ - v_e^-$ and extending linearly. The *1-cycles* of G are the elements of $H_1(G, \mathbb{R}) = \ker(\partial)$. The group $H_1(G, \mathbb{Z})$ is the intersection of $H_1(G, \mathbb{R})$ with $C_1(G, \mathbb{Z})$.

Dually, one may define *1-forms* on G as elements of the real vector space spanned by $\{d_e : e \in E(G)\}$. There is a "discrete intergration" on 1-forms defined as follows. Choose a basis element de, and an edge e'. Define

$$\int_{e'} de = \begin{cases} \ell(e) & \text{if } e' = e \\ 0 & \text{if } e' \neq e \end{cases}.$$

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By extending linearly, this determines $\int_\alpha \omega$ for arbitrary 1-chains. Among the 1-forms are certain distinguished forms known as the *harmonic* 1-forms. A 1-form $\omega = \sum \omega_e d_e$ is *harmonic* if for all $v \in V(G)$, we have

$$\sum_{\substack{e \in E(G) \\ v = v_e^+}} \omega_e = \sum_{\substack{e \in E(G) \\ v = v_e^-}} \omega_e.$$

Let $\Omega(G)$ denote the vector space of harmonic 1-forms. Integration on $\Omega(G)$ gives a pairing

$$\langle\ ,\ \rangle_\ell : \Omega(G) \times C_1(G, \mathbb{R}) \to \mathbb{R}$$

by $\langle \omega, \alpha \rangle_\ell = \int_\alpha \omega$, which depends on ℓ. When restricted to $\Omega(G) \times H_1(G, \mathbb{R})$, this gives a perfect pairing. In other words, letting $\Omega^*(G)$ denote the dual of $\Omega(G)$, we have the following.

Lemma 9.1. *The map $\alpha \mapsto \langle\ , \alpha \rangle_\ell$ gives an isomorphism*

$$H_1(G, \mathbb{R}) \xrightarrow{\sim} \Omega^*(G).$$

Proof. Suppose we have a 1-form $\omega = \sum \omega_e d_e$, let $\alpha = \sum \omega_e e$. Note that ω is harmonic if and only if α is a 1-cycle. This implies that the map $\omega \mapsto \alpha$ gives an isomorphism $H_1(G, \mathbb{R}) \cong \Omega^*(G)$. In particular, $\dim(H_1(G, \mathbb{R})) = \dim(\Omega^*(G))$. As a result of this, it suffices to show that the map $H_1(G, \mathbb{R}) \to \Omega^*(G)$ given by $\alpha \mapsto \int_\alpha$ has trivial kernel. Suppose we are given a 1-cyle α such that $\int_\alpha = 0$. Write α as $\sum a_e e$, and let $\omega = \sum a_e d_e$. Then $0 = \langle \omega, \alpha \rangle = \sum a_e^2 \ell(e)$. Since $\ell(e) > 0$ for all edges e, we have $a_e = 0$, and thus $\alpha = 0$. It follows that the kernel is trivial. □

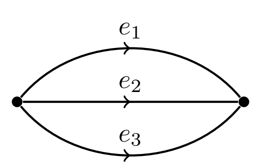
9.2. **Model Jacobians.** Let α now be a 1-chain with integral coefficients, then $\alpha \mapsto \int_\alpha$ gives a map $C_1(G, \mathbb{Z}) \to \Omega^*(G)$. Let $\Omega^\sharp(G)$ denote the image. Then, since the integration pairing induces an isomorphism from $H_1(G, \mathbb{Z})$ to its image in $\Omega^*(G)$, we can view $H_1(G, \mathbb{Z})$ as a subgroup of $\Omega^*(G)$ contained in $\Omega^\sharp(G)$.

Definition 9.2. *The* model Jacobian *of G is the quotient*

$$J_\ell(G) = J(G, \ell) := \Omega^\sharp(G)/H_1(G, \mathbb{Z}).$$

The notation and terminology are intended to highlight the dependence of this group on the metric model of the underlying combinatorial graph, given by the length function ℓ.

Example 9.3. Let G be the following graph, with orientation given by the arrows.



Assume that all edge lengths are equal to 1. The cycles $c_1 = e_1 - e_2$ and $c_2 = e_3 - e_2$, form a minimal generating set for $H_1(G, \mathbb{Z})$. The group $\Omega^\sharp(G)$ is generated by $\int_{e_1}$ and $\int_{-e_3}$. The model Jacobian of G is isomorphic to $\mathbb{Z}/3\mathbb{Z}$: the elements of $J_\ell(G)$ correspond to the classes $[0]$, $[\int_{e_1}]$ and $[\int_{-e_3}]$ in $\Omega^\sharp(G)/H_1(G, \mathbb{Z})$.

Remark 9.4. The model Jacobian is not necessarily a finite group. For instance, one can observe that if the edge lengths of G are $\mathbb{Q}$-independent elements of $\mathbb{R}$, then $J_\ell(G)$ is a free abelian group of finite rank.

9.3. **Discrete Abel-Jacobi.** In Example 9.3, the degree 0 Picard group of G (viewed as a discrete graph) is also equal to $\mathbb{Z}/3\mathbb{Z}$. As we will see, this is not a coincidence.

The *model divisor group* $\mathrm{Div}_\ell(G)$ of G is the group $C_0(G,\mathbb{Z})$ of 0-chains. We define $\mathrm{Div}^0_\ell(G)$ as the subgroup

$$\mathrm{Div}^0_\ell(G) := \{\sum_{v\in V(G)} a_v v \in C_0(G,\mathbb{Z}) : \sum_{v\in V(G)} a_v = 0\}.$$

Consider the operator $d : C_0(G,\mathbb{R}) \to C_1(G,\mathbb{R})$, taking $f \in C_0(G,\mathbb{R})$ to the 1-chain df defined by

$$df(e) := \frac{f(v_e^+) - f(v_e^-)}{\ell(e)} \text{ for all } e \in E(G).$$

Let $\mathrm{Im}(d)_\mathbb{Z} = \mathrm{Im}(d) \cap C_1(G,\mathbb{Z})$. The *model principal divisor group* $\mathrm{Prin}_\ell(G)$ is the group $\partial(\mathrm{Im}(d)_\mathbb{Z})$.

Definition 9.5. *The* model Picard group *of degree* 0 $\mathrm{Pic}^0_\ell(G)$ *of* G *is the quotient*

$$\mathrm{Pic}^0_\ell(G) := \mathrm{Div}^0_\ell(G)/\,\mathrm{Prin}_\ell(G).$$

In complex geometry, the Abel-Jacobi theorem states that, for a smooth compact Riemann surface X, there is a complex analytic isomorphism between the groups $\mathrm{Pic}^0(X)$ of complex line bundles on X, and the Jacobian $J(X)$. Recall that $J(X)$ is the quotient $\Omega^*(X)/H_1(X,\mathbb{Z})$, where $\Omega^*(X)$ is the dual of the vector space of global holomorphic forms on X. In analogy, the following is a "discretized" Abel-Jacobi theorem.

Theorem 9.6. *There is a natural isomorphism* $J_\ell(G) \xrightarrow{\sim} \mathrm{Pic}^0_\ell(G)$ *induced by the map* $\partial : C_1(G,\mathbb{Z}) \to \mathrm{Div}^0_\ell(G)$.

Remark 9.7. Given an edge e. View e as a 1-chain, then $\partial(e)$ is an element of $\mathrm{Div}^0_\ell(G)$. Thus, the image of $\partial : C_1(G,\mathbb{Z}) \to C_0(G,\mathbb{Z})$ is contained in $\mathrm{Div}^0_\ell(G)$.

In order to establish the theorem, we require the following result.

Lemma 9.8. *The kernel of* d *is the constant functions, and we have the decomposition*

$$C_1(G,\mathbb{R}) = H_1(G,\mathbb{R}) \oplus \mathrm{Im}(d).$$

Moreover, the kernel of $C_1(G,\mathbb{R}) \to \Omega^*(G)$ *given by integration is equal to* $\mathrm{Im}(d)$.

Proof. Choose $f \in C_0(G,\mathbb{R})$, and suppose $df \neq 0$. There is a vertex v of G such that f achieves a maximum at v. Given an edge e such that $v_e^- = v$, we have

$$-\left(\frac{f(v_e^+) - f(v_e^-)}{\ell(e)}\right) \geq 0.$$

Similarly, if $v_e^+ = v$, then

$$\frac{f(v_e^+) - f(v_e^-)}{\ell(e)} \geq 0$$

which implies that $\partial df(v) \geq 0$.

Without loss of generality, since f can be taken to be non-constant, we can assume that v has an adjacent vertex v' such that $f(v') < f(v)$. This implies that $\partial df(v) > 0$. Thus $\partial df \neq 0$, which implies that ∂df is not in $H_1(G,\mathbb{Z})$, and $H_1(G,\mathbb{R}) \cap \operatorname{Im}(d) = \{0\}$.

The same argument forces that the kernel of d is precisely the constant functions: if f is not constant then there is pair of adjacent vertices v and v' such that $f(v) > f(v')$. In particular, $\dim(\operatorname{Im}(d)) = \dim(C_0(G,\mathbb{R})) - 1$. By an Euler characteristic calculation,

$$\dim(C_1(G,\mathbb{R})) = \dim(H_1(G,\mathbb{R})) + \dim(C_0(G,\mathbb{R})) - 1,$$

establishing the first part of the lemma.

For the second part, since $\dim(H_1(G,\mathbb{R}) = \dim(\Omega^*(G))$, it suffices to show that the kernel is contained in $\operatorname{Im}(d)$. Let ω be a harmonic 1-form. Writing $\omega = \sum \omega_e de$, we have

$$\int_{df} \omega = \sum_{e\in E(G)} \sum_{e'\in E(G)} \frac{f(e^+) - f(e^-)}{\ell(e)} \omega_{e'} \int_e de' = \sum_{e\in E(G)} [f(e^+) - f(e^-)]\omega_e$$

$$= \sum_{x\in V(G)} f(x) \left(\sum_{\substack{e\in E(G) \\ v=v_e^+}} \omega_e - \sum_{\substack{e\in E(G) \\ v=v_e^-}} \omega_e \right)$$

and the last expression is zero since ω is harmonic. □

Proof of Theorem 9.6. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(G,\mathbb{Z}) & \longrightarrow & H_1(G,\mathbb{Z}) \oplus \operatorname{Im}(d)_{\mathbb{Z}} & \xrightarrow{\partial} & \operatorname{Prin}_\ell(G) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(G,\mathbb{Z}) & \longrightarrow & C_1(G,\mathbb{Z}) & \xrightarrow{\partial} & \operatorname{Div}^0_\ell(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \int_\alpha} & & \downarrow & & \\
 & & 0 & & J_\ell(G) & \dashrightarrow & \operatorname{Pic}^0_\ell(G) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

We claim that the entire diagram, minus the dotted arrow from $J_\ell(G)$ to $\operatorname{Pic}^0_\ell(G)$, is exact. By definition, the top row and the right column are exact. Exactness of the middle column follows from Lemma 9.8.

For the second row, note that $\operatorname{Div}^0_\ell(G)$ is equal to $C_0(G,\mathbb{Z})$ modulo the constant functions. By Lemma 9.8, $C_1 = \ker(d) \oplus \operatorname{Im}(d)$, and $\operatorname{Im}(d)$ is equal to $C_0(G,\mathbb{R})$ modulo constant functions. Exactness for the second row then follows.

We thus have an induced map from $J_\ell(G)$ to $\operatorname{Pic}^0_\ell(G)$, and by the Snake Lemma (or a straightforward diagram chase) it is an isomorphism. □

In the edge lengths of G are equal, then it is easy to see that $\mathrm{Pic}^0_\ell(G) \cong \mathrm{Pic}^0(G)$. We thus have the following corollary, which was hinted at the beginning of this subsection.

Corollary 9.9. *If the edge lengths of G are equal, then we have an isomorphism*

$$\mathrm{Pic}^0(G) \xrightarrow{\sim} J_\ell(G).$$

9.4. **Tropical Abel-Jacobi.** Intuitively speaking, since G is a discrete representation of a metric graph Γ, $\mathrm{Pic}^0_\ell(G)$ and $J_\ell(G)$ can be thought of as discrete approximations of respectively $\mathrm{Pic}^0(\Gamma)$ and $J(\Gamma)$. We now make this intuition precise, and use Theorem 9.6, together with a compatibility observation for refinements of graphs, to prove the tropical Abel-Jacobi theorem.

Let G' be a refinement of G. Any such refinement inherits an orientation from G in the obvious manner. Since $V(G) \subset V(G')$, we have a natural inclusion $\mathrm{Div}_\ell(G) \subset \mathrm{Div}_\ell(G')$. Also, since $\mathrm{Div}(\Gamma)$ is equal to the subset of $\mathbb{Z}^\Gamma$ with finite support, we get an inclusion $\mathrm{Div}_\ell(G') \subset \mathrm{Div}(\Gamma)$. It follows from definitions thatif G'' refines G', which in turn refines G, then there is a sequence of inclusions,

$$\mathrm{Pic}^0_\ell(G) \hookrightarrow \mathrm{Pic}^0_\ell(G') \hookrightarrow \mathrm{Pic}^0_\ell(G'') \hookrightarrow \mathrm{Pic}^0(\Gamma).$$

Consider the directed system

$$\{\mathrm{Pic}^0_\ell(G') \hookrightarrow \mathrm{Pic}^0_\ell(G'') : \ G'' \text{ refines } G' \text{ refines } G\}.$$

Proposition 9.10. *The injections $\mathrm{Pic}^0_\ell(G') \hookrightarrow \mathrm{Pic}^0(\Gamma)$ induce an isomorphism*

$$\varinjlim_{G' \text{ refines } G} \mathrm{Pic}^0_\ell(G') \xrightarrow{\sim} \mathrm{Pic}^0(\Gamma).$$

Proof. Injectivity follows from left-exactness of direct limit functor. Thus it suffices to show surjectivity. Choose a divisor class $[D] \in \mathrm{Pic}^0(\Gamma)$, and choose a representative $D \in [D]$. Then, to show surjectivity, it suffices to show that there is a refinement G' of G and a divisor $D' \in \mathrm{Div}^0_\ell(G')$ such that the map $\mathrm{Div}^0_\ell(G') \to \mathrm{Div}^0(\Gamma)$ sends D' to D.

Write D as a formal sum of points $x_1 + \cdots + x_n$, where $x_i \in \Gamma$ for $i \in 1, ..., n$. Choose G' such that $x_i \in V(G)$ for all i. Then, the divisor $\sum_i x_i \in \mathrm{Div}_\ell(G')$ is the desired D'. □

By mapping an edge $e \in E(G)$ to the formal sum of the edges in $E(G')$ that e is subdivded into, we obtain an injection $C_1(G, \mathbb{Z}) \hookrightarrow C_1(G', \mathbb{Z})$. This restricts to an isomorphism $H_1(G, \mathbb{Z}) \xrightarrow{\sim} H_1(G', \mathbb{Z})$. It is straightforward to check that this induces an inclusion $J_\ell(G) \hookrightarrow J_\ell(G')$.

In similar fashion as above, there is an isomorphism $H_1(G, \mathbb{R}) \xrightarrow{\sim} H_1(G', \mathbb{R})$, which implies that there is an isomorphism

$$\frac{\Omega^*(G)}{H_1(G, \mathbb{Z})} \xrightarrow{\sim} \frac{\Omega^*(G')}{H_1(G', \mathbb{Z})}$$

This isomorphism is compatible with the inclusions $J_\ell(G) \hookrightarrow J_\ell(G')$. In other words, given G'' refining G' refining G, the following diagram commutes.

$$\begin{array}{ccccc} J_\ell(G) & \hookrightarrow & J_\ell(G') & \hookrightarrow & J_\ell(G'') \\ \downarrow & & \downarrow & & \downarrow \\ \dfrac{\Omega^*(G)}{H_1(G,\mathbb{Z})} & \hookrightarrow\!\!\!\twoheadrightarrow & \dfrac{\Omega^*(G')}{H_1(G',\mathbb{Z})} & \hookrightarrow\!\!\!\twoheadrightarrow & \dfrac{\Omega^*(G'')}{H_1(G'',\mathbb{Z})} \end{array}$$

Recall that we have a canonical isomorphism $J(\Gamma) \cong \dfrac{\Omega^*(G)}{H_1(G,\mathbb{Z})}$. Therefore we have the sequence of inclusions

$$J_\ell(G) \hookrightarrow J_\ell(G') \hookrightarrow J_\ell(G'') \hookrightarrow J(\Gamma).$$

Consider the directed system

$$\{J_\ell(G') \hookrightarrow J_\ell(G'') : \ G'' \text{ refines } G' \text{ refines } G\}.$$

Proposition 9.11. *The injections $J_\ell(G') \hookrightarrow J(\Gamma)$ induce an isomorphism*

$$\varinjlim_{G' \text{ refines } G} J_\ell(G') \xrightarrow{\sim} J(\Gamma).$$

Proof. Again, it suffices to show surjectivity. Choose an element ϕ in $\Omega^*(G)$, and write it as $\int_\alpha$ for some $\alpha \in H_1(G,\mathbb{R})$. The isomorphism $H_1(G,\mathbb{R}) \cong H_1(G',\mathbb{R})$ allows us to view $\int_\alpha$ as an element of $\Omega^*(G')$. To show surjectivity, it suffices to show that there is an element $\beta \in C_1(G',\mathbb{Z})$ such that $\int_\beta$ is equal to $\int_\alpha$.

First, we give an explicit description of α. Choose a spanning tree T of G, and let $e_1,...,e_g$ denote the edges in $G-T$. We obtain a cycle α_j by adjoining e_j with the unique path in T from v_e^+ to v_e^-. Then $\{\int_{\alpha_j}\}$ forms a basis for $\Omega^*(G)$, and we can write α as $\sum_j t_j \int_{\alpha_j}$.

We now construct G' and β. First, write α_j as $\sum \alpha_j(e)e$, and let $\omega_j = \sum_j \alpha_j(e)de$. For each j, choose n_j and $0 < u_j < \ell(e_j)$ such that $n_j u_j = t_j \int_{\alpha_j} \omega_j$. Pick the point p_j on e_j such that $\mathrm{dist}(v_{e_j}^-, p_j) = u_j$. Take G' such that $V(G')$ is equal to $V(G)$ plus the points p_j. Set $e_j' = [v_{e_j}^-, p_j]$. Then, one can check that $\beta = \sum n_j e_j'$ is our desired 1-chain. □

Now, we derive the tropical Abel-Jacobi theorem from the previous two propositions and Theorem 9.6.

Theorem 9.12. *The isomorphism $\mathrm{Pic}_\ell^0(G') \cong J_\ell(G')$ induces an isomorphism*

$$\varinjlim_{G' \text{ refines } G} \mathrm{Pic}_\ell^0(G') \cong \varinjlim_{G' \text{ refines } G} J_\ell(G').$$

In particular, this gives an isomorphism $\mathrm{Pic}^0(\Gamma) \cong J(\Gamma)$.

Proof. Immediate from Theorem 9.6 and Propositions 9.10 and 9.11. □

9.4.1. *The tropical Abel-Jacobi map.* Choose a basepoint $p \in \Gamma$. For any $p' \in \Gamma$, we have the divisor $D_{p'} = p' - p \in \mathrm{Div}^0(\Gamma)$. By sending p' to the class $[D_{p'}]$ in $\mathrm{Pic}^0(\Gamma)$, and by identifying $\mathrm{Pic}^0(\Gamma)$ with $J(\Gamma)$, we obtain a map

$$\alpha_p : \Gamma \to J(\Gamma)$$

which depends on the choice of basepoint p. This map is continuous, and the induced map $\pi_1(\Gamma,p) \to \pi_1(J(\Gamma),0)$ is precisely the Hurewicz map $\pi_1(\Gamma,p) \to$

$H_1(\Gamma, \mathbb{Z})$. The map α_p is referred as the *tropical Abel-Jacobi map* based at p and will make an appearance in later lectures.

References

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