WEIGHT FUNCTIONS AND ESSENTIAL SKELETA

MIRCEA MUSTAŢĂ

These are notes based on the author's talk at the Simons Symposium on Non-Archimedean and Tropical Geometry, Puerto Rico, February 1-7, 2015. The goal is to describe certain functions defined on analytifications of complete varieties over $k((t))$ and the skeleta that are defined using these functions, following the author's joint work with Johannes Nicaise [MN]. The notions that we discuss parallel familiar ones in the study of singularities in birational geometry.

1. The birational geometry setting

Almost everything here is classical and well-known to people studying singularities in birational geometry. However, the point of view that is taken, based on non-Archimedean geometry, is due to Boucksom, Favre, and Jonsson [BFJ] (see also [JM]).

Let X be a smooth variety over an algebraically closed field k, with $\mathrm{char}(k) = 0$. In general, one wants to allow X to have mild singularities, but we do not need here this more general framework.

We want to study the singularities of a given hypersurface H in X. This is done by comparing two invariants, for all

$$v \in \mathrm{Val}_X := \{\text{nonzero real valuations of } k(X) \text{ with center on } X\}, \text{ namely}$$

• The first invariant is $v(H)$;

• The second invariant is the log discrepancy $A_X(v)$, defined below. This only depends on v and it is used in order to normalize $v(H)$ (which can be arbitrarily large).

Remark. The classical point of view is to only consider divisorial valuations, that is, valuations of the form $v = \mathrm{ord}_E$, where E is a prime divisor on some normal variety Y, that has a proper birational morphism $Y \longrightarrow X$. It was noticed in [BFJ] that one can look at all valuations and not just at divisorial ones. This does not add anything when one studies the singularities of a hypersurface, but it is very useful when studying singularities of graded sequences of ideals or plurisubharmonic functions (see [BFJ] and [JM]). The advantage is that by considering all valuations, one can make use of compactness arguments.

We now recall the definition of the log discrepancy function $A_X \colon \mathrm{Val}_X \longrightarrow \mathbf{R}_{\geq 0} \cup \{\infty\}$.

Step 1. If v is a divisorial valuation, then we can find a proper birational morphism $f \colon Y \longrightarrow X$, with Y normal, and a prime divisor E on Y such that $v = \mathrm{ord}_E$. In this case, we take

$$A_X(v) = 1 + \mathrm{ord}_E(K_{Y/X}),$$

where $K_{Y/X}$ is the effective divisor on Y locally defined by the Jacobian of f.

Step 2. Suppose now that v is a *quasi-monomial* valuation. This means that we can find a proper, birational morphism $f\colon Y \longrightarrow X$, with Y smooth, a not-necessarily-closed point $p \in Y$, a system of coordinates $y_1, \ldots, y_r$ at p, and $a_1, \ldots, a_r \in \mathbf{R}_{\geq 0}$ (not all 0) such that v is the monomial valuation, in this system of coordinates, with $v(y_i) = a_i$. More precisely, v is the unique valuation such that for every $f = \sum_{u \in \mathbf{N}^r} c_u y^u$, with $c_u \in k$, we have

$$v(f) = \min \left\{ \sum_{i=1}^{r} c_i a_i \mid c_u \neq 0 \right\}.$$

In this case, we put

$$A_X(v) = \sum_{i=1}^{r} a_i \cdot A_X(\mathrm{ord}_{E_i}),$$

where E_i is the divisor defined by (y_i). Note that the a_i are all rational numbers if and only if we can write $v = q \cdot \mathrm{ord}_E$ for some divisorial valuation ord_E and in this case the definition gives $A_X(v) = q \cdot A_X(\mathrm{ord}_E)$.

Step 3. Suppose now that $v \in \mathrm{Val}_X$ is arbitrary. For every $f\colon Y \longrightarrow X$ proper, birational, with Y smooth, if y is the center $c_Y(v)$ of v, and $y_1, \ldots, y_r$ is a system of coordinates at p, we define the valuation $\rho_Y(v)$ to be the quasi-monomial valuation corresponding to this data and such that $\rho_Y(v) = v$ on each y_i. One defines

$$A_X(v) := \sup_{Y/X} A(\rho_Y(v)) \in \mathbf{R}_{\geq 0} \cup \{\infty\},$$

where the supremum is over all choices of $Y \longrightarrow X$ and over all possible systems of coordinates at $c_Y(v)$.

We have the following basic properties (see [JM] for proofs):

• If v is a quasi-monomial valuation, then the general definition of $A_X(v)$ agrees with the one in Step 2.

• In fact, given $f\colon Y \longrightarrow X$ and a system of coordinates as above, we have $A_X(v) \geq A_X(\rho_Y(v))$, with equality if and only if v is quasi-monomial with respect to f and this system of coordinates.

• The function $A_X\colon \mathrm{Val}_X \longrightarrow \mathbf{R}_{\geq 0} \cup \{\infty\}$ is lower semicontinuous.

Suppose that $f\colon Y \longrightarrow X$ is a morphism as above and D is a simple normal crossing divisor on X. If $\mathrm{QM}(Y, D) \subseteq \mathrm{Val}_X$ is the set of all quasi-monomial valuations with respect to the systems of coordinates given by the components of D, then $\mathrm{QM}(Y, D)$ has an "integral cone complex" structure. By definition, A_X is piecewise linear, continuous, and integeral on $\mathrm{QM}(Y, D)$.

Remark. It is sometimes convenient to take the quotient of $\mathrm{QM}(Y, D)$ by the obvious $\mathbf{R}_{>0}$-action, in order to get a simplicial complex (see [BFJ]).

We now return to the hypersurface H in X. The *log canonical threshold* of (X,H) is

$$\mathrm{lct}(X,H)=\inf_{v\in \mathrm{Val}_X}\frac{A_X(v)}{v(H)}.$$

It is a basic fact that if $f\colon Y\longrightarrow X$ is a log resolution of (X,H) (that is, f is proper and birational, X is smooth, and $f^*(H)+K_{Y/X}$ has simple normal crossings, then the infimum in the definition of $\mathrm{lct}(X,H)$ is achieved by some $v=\mathrm{ord}_E$, where E is a prime divisor on X. This follows from the fact that if w is any valuation and $w'=\rho_Y(w)$ (with respect to the simple normal crossing divisor $D=f^*(H)+K_{Y/X}$), then $w'(H)=w(H)$ and $A_X(w)\geq A_X(w')$. In fact, this shows more: for every $w\in \mathrm{Val}_X$ that achieves the minimum in the definition of $\mathrm{lct}(X,H)$, we see that $w\in \mathrm{QM}(Y,D)$.

The divisors that compute $\mathrm{lct}(X,H)$ (that is, the ones that achieve the minimum in the definition of $\mathrm{lct}(X,H)$) play an important role in birational geometry. The union of their images in X is the *non-klt* locus $\mathrm{Nklt}(X,cH)$, where $c=\mathrm{lct}(X,H)$. The following theorem is a very useful result in birational geometry, that can be used to study the properties of this locus.

Theorem (Kollár, Shokurov). Given any log resolution $f\colon Y\longrightarrow X$ of (X,H), the union $\bigcup_E E$, where the union is over the primes divisors E on Y that compute $\mathrm{lct}(X,H)$, is connected in the neighborhood of any fiber of f.

Remark. One can also reformulate this result in terms of a suitable subcomplex of the intersection complex of the simple normal crossing divisor $f^*(H)+K_{Y/X}$.

2. The non-Archimedean setting

All new results here are based on the joint paper [MN] with Johannes Nicaise. Let $K=k((t))$, where k is an algebraically closed field of characteristic 0, and let $R=k[[t]]$. Certain results carry over to positive characteristic if we work with varieties whose models are known to admit log resolutions (for example, curves), but we will not add here this extra level of complexity.

Let X be a smooth, projective, geometrically connected variety over K. The first goal is to define in this setting an analogue of the log discrepancy function. This will be a function $X^{\mathrm{an}}\longrightarrow \mathbf{R}_{\geq 0}\cup\{\infty\}$, where X^{an} is the Berkovich analytification of X. This function will depend on the choice of some nonzero $\omega\in\Gamma(X,\omega_X^{\otimes m})$, with $m\geq 1$.

The birational models $Y\longrightarrow X$ we considered before are replaced in this setting by SNC models $\mathcal{X}$ of X/K, that is, regular, projective schemes over $\mathrm{Spec}(R)$, whose special fiber $\mathcal{X}_k$ is a simple normal crossing divisor on $\mathcal{X}$, and whose generic fiber is isomorphic to X. Note that existence of such models is guaranteed in our setting by Hironaka's theorem.

Suppose now that we fix a nonzero $\omega\in\Gamma(X,\omega_X^{\otimes m})$, with $m\geq 1$. We describe in three steps how to define the *weight function* $\mathrm{wt}_\omega\colon X^{\mathrm{an}}\longrightarrow \mathbf{R}_{\geq 0}\cup\{\infty\}$ associated to ω.

Step 1. Suppose that $x\in X^{\mathrm{an}}$ is a divisorial point, that is, there is a model $\mathcal{X}$ as above and an irreducible component E of the special fiber $\mathcal{X}_k$ such that $x=\frac{1}{N}\mathrm{ord}_E$, where

$N = \mathrm{ord}_E(\mathcal{X}_k)$. Consider now the relative dualizing sheaf $\omega_{\mathcal{X}/R}$, which is a line bundle on $\mathcal{X}$ by the assumption that $\mathcal{X}$ is regular. We may consider ω as a rational section of $\omega_{\mathcal{X}/R}$. As such, it determines a Cartier divisor $\mathrm{div}_{\mathcal{X}}(\omega)$. If f is a local equation for $\mathrm{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{\mathrm{red}}$ at the generic point of E, then we put

$$\mathrm{wt}_\omega(x) = \frac{1}{N}\,\mathrm{ord}_E(f).$$

It is straightforward to see that the definition only depends on x and not on the model $\mathcal{X}$.

Step 2. We now extend the above definition to the case when $x \in X^{\mathrm{an}}$ is a monomial point. Suppose that $\mathcal{X}$ is an SNC model of X as above. A monomial point corresponds to a connected component of $E_1 \cap \ldots \cap E_r$, for some irreducible components $E_1, \ldots, E_r$ of $\mathcal{X}_k$ and to some $a_1, \ldots, a_r \in \mathbf{R}_{\geq 0}$ such that $\sum_{i=1}^r a_i N_i = 1$ (here $N_i = \mathrm{ord}_{E_i}(\mathcal{X}_k)$). If the y_i are the local equations of the E_i at the generic point ξ of $E_1 \cap \ldots \cap E_r$ and if $f = \sum_{u \in \mathbf{N}^r} c_u y^u$, with each $c_u \in \widehat{\mathcal{O}_{\mathcal{X},\xi}}$ either zero or invertible, then the valuation v_x corresponding to x satisfies

$$v_x(f) = \min\left\{\sum_{i=1}^r u_i a_i \mid c_u \neq 0\right\}.$$

As before, if f is a local equation for $\mathrm{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{\mathrm{red}}$ at ξ, then we put

$$\mathrm{wt}_\omega(x) = v_x(f).$$

It is clear that when x is a divisorial point, this definition agrees with the one in Step 1.

Before we proceed with the general case, we recall some basic facts, due to Berkovich, concerning monomial points.

• The *Berkovich skeleton* $\mathrm{Sk}(\mathcal{X})$ associated to the model $\mathcal{X}$ is the subset of X^{an} consisting of all monomial point associated to $\mathcal{X}$. This is homeomorphic to the intersection complex $\Delta(\mathcal{X}_k)$ of the special fiber.

• For every SNC model $\mathcal{X}$, there is a canonical retraction $\rho_{\mathcal{X}} \colon X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathcal{X})$ and there is a homeomorphism

$$X^{\mathrm{an}} \simeq \varprojlim_{\mathcal{X}} \mathrm{Sk}(\mathcal{X}).$$

• For every SNC model $\mathcal{X}$, the Berkovich skeleton $\mathrm{Sk}(\mathcal{X})$ is a strong deformation retract of X^{an}.

We now go back to the definition of the weight function.

Step 3. Using the canonical retractions $X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathcal{X})$, we extend the definition of the weight function, by putting

$$\mathrm{wt}_\omega(x) := \sup_{\mathcal{X}} \mathrm{wt}_\omega(\rho_{\mathcal{X}}(x)).$$

The weight function satisfies the following properties (see [MN] for proofs):

• If $\mathcal{X}$ and $\mathcal{X}'$ are SNC models of X such that we have a morphism $\mathcal{X}' \longrightarrow \mathcal{X}$ over R inducing the identity on X, then

$$\mathrm{wt}_\omega(\rho_{\mathcal{X}}(x)) \leq \mathrm{wt}_\omega(\rho_{\mathcal{X}'}(x)),$$

with equality if and only if $x \in \mathrm{Sk}(\mathcal{X})$. In particular, this implies that the general definition in Step 3 agrees in the case of monomial points with the definition in Step 2.

• The function wt_ω is lower semicontinuous on X^{an}.

We can now define an analogue of the log canonical threshold in this setting, namely

$$\mathrm{wt}_\omega(X) := \inf\{\mathrm{wt}_\omega(x) \mid x \in X^{\mathrm{an}}\}.$$

As in the case of the log canonical threshold, one can show that the infimum in the definition of $\mathrm{wt}_\omega(X)$ is achieved on every SNC model $\mathcal{X}$.

The *Kontsevich-Soibelman skeleton* is

$$\mathrm{Sk}(X,\omega) := \{x \in X^{\mathrm{an}} \mid \mathrm{wt}_\omega(x) = \mathrm{wt}_\omega(X)\}.$$

As in the case of the log canonical threshold, for every SNC model $\mathcal{X}$, we have $\mathrm{Sk}(X,\omega) \subseteq \mathrm{Sk}(\mathcal{X})$.

Of course, the Kontsevich-Soibelman skeleton depends on the choice of pluricanonical form ω. An important special case, studied by Kontsevich and Soibelman [KS], is that when X is a Calabi-Yau variety and ω is taken to be a volume form. For two different choices of ω the value of $\mathrm{wt}_\omega(X)$ differs by an integer, but the skeleta $\mathrm{Sk}(X,\omega)$ are independent of ω.

For an arbitrary variety X (with the property that $h^0(X,\omega_X^{\otimes m}) \neq 0$ for some $m \geq 1$), in order to eliminate the dependence on ω, we define the *essential skeleton*

$$\mathrm{Sk}(X) := \bigcup_{\omega} \mathrm{Sk}(X,\omega),$$

where the union is over all nonzero $\omega \in \Gamma(X,\omega_X^{\otimes m})$, with $m \geq 1$. This invariant of X seems to be particularly well-behaved when some multiple of ω_X is globally generated.

Example (Baker-Nicaise). If X is a curve of genus at least 1 with semistable reduction, then $\mathrm{Sk}(X) = \mathrm{Sk}(\mathcal{X})$, where $\mathcal{X}$ is the minimal SNC model.

We prove the following analogue of the connectedness result due to Kollár and Shokurov.

Theorem. If $h^0(X,\omega_X) = 1$ and $\omega \in \Gamma(X,\omega_X)$ is nonzero, then $\mathrm{Sk}(X,\omega)$ is connected.

The proof of the theorem makes use of vanishing theorems, in a slightly different way than in Kollár's proof. Since we are not in the setting of schemes of finite type over k, we need to deduce the statements of the vanishing theorems that we need (especially, Kollár's torsion-freeness theorem) from those in the classical settings.

A much stronger result, making use of the techniques of the Minimal Model Program, was subsequently obtained by Nicaise and Xu [NX] (see the talk of Chenyang Xu).

Remark. Halle and Nicaise [HN] have developed several other analogies between $\mathrm{wt}_\omega(X)$ and the log canonical threshold, by relating $\mathrm{wt}_\omega(X)$ with other invariants that come up in this non-Archimedean setting, such as poles of motivic zeta functions, monodromy, etc. A case that is particularly well-understood is that of Abelian varieties, see [HN].

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