POLES OF MAXIMAL ORDER OF IGUSA ZETA FUNCTIONS

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Abstract. These are notes for the third in a series of lectures at the 2015 Simons Symposium on Tropical and Non-Archimedean Geometry. The first two lectures were given by Mircea Mustaţă and Chenyang Xu; their content is mostly covered by the survey article "Berkovich skeleta and birational geometry" that will appear in the proceedings of the 2013 Symposium. Here we discuss an application to Igusa zeta functions, obtained in collaboration with Chenyang Xu and published in the article "Poles of maximal order of motivic zeta functions" (arXiv:1403.6792, to appear in Duke Math. J.). The present notes can be read independently of the aforementioned survey article.

1. Igusa's p-adic zeta functions

(1.1) We fix a non-constant polynomial f in $\mathbb{Z}[x_1, \ldots, x_n]$, for some $n \geq 1$, with $f(0) = 0$. For every prime number p and every integer $d \geq 0$, we set

$$N_{f,p}(d) = \sharp\{x \in (p\mathbb{Z}/p^{d+1}\mathbb{Z})^n \mid f(x) \equiv 0 \mod p^{d+1}\}.$$

Thus $N_{f,p}(d)$ is the number of solutions of the congruence $f \equiv 0$ modulo p^{d+1} that reduce to the origin modulo p. To study the asymptotic behaviour of these numbers as $d \to \infty$, we introduce the generating series

$$P_{f,p}(T) = \sum_{d \geq 0} N_{f,p}(d)T^d \quad \in \mathbb{Z}[[T]].$$

Now it is natural to ask the following question.

Question (Borevich-Shafarevich, 1966)**.** Is $P_{f,p}(T)$ a rational function?

(1.2) Note that the series $P_{f,p}(T)$ is easy to compute if the zero locus of f in $\mathbb{A}^n_{\mathbb{Z}}$ is smooth over $\mathbb{Z}$ at the origin of $\mathbb{A}^n_{\mathbb{F}_p}$: by Hensel's lemma, we have

$$P_{f,p}(T) = \frac{1}{1 - p^{n-1}T}.$$

However, the computation is substantially more difficult if f has a singularity, already in the case of the plane cusp $f = (x_1)^2 - (x_2)^3$. We can expect that $P_{f,p}(T)$ will reflect some interesting properties of the singularity.

(1.3) Igusa has proven in 1975 that $P_{f,p}(T)$ is always rational. A key step in the proof is to rewrite the generating series $P_{f,p}(T)$ as a p-adic integral

$$Z_{f,p}(s) = \int_{(p\mathbb{Z}_p)^n} |f|_p^s |dx|$$

where s is a complex variable and $|dx|$ is the Haar measure on $\mathbb{Z}_p^n$. This p-adic integral converges $Z_{f,p}(s)$ if $\Re(s) > 0$ and defines a complex analytic function on the right half-plane, which is called Igusa's p-adic zeta function. It is an easy exercise to express $Z_{f,p}(s)$ in terms of $P_{f,p}(p^{-s})$, so that it is enough to prove that $Z_{f,p}(s)$ is a rational function in p^{-s} (and, in particular, extends to a meromorphic function on $\mathbb{C}$). Igusa proved this by taking a resolution of singularities for f over $\mathbb{Q}$ and using the change of variables formula for p-adic integrals to reduce to the case where f is a monomial, which can be solved by a simple computation.

(1.4) This proof not only establishes the rationality of $Z_{f,p}(s)$ but also provides some interesting information about the possible poles of $Z_{f,p}(s)$ and their expected orders in terms of the geometry of a resolution of singularities for f. This is a striking result, because it relates arithmetic properties of f (the poles of the zeta function) with geometric properties of f (the geometry of a resolution of singularities). A completely explicit formula for $Z_{f,p}(s)$ in terms of a resolution of f was later given by Denef, for $p \gg 0$. In the next section, we will review the precise formulation of Igusa's theorem and Denef's formula.

2. Denef's formula

(2.1) First, we need to introduce some notation. Let $h : Y \to \mathbb{A}^n_{\mathbb{Q}}$ be a log-resolution for the morphism $f : \mathbb{A}^n_{\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}}$ defined by the polynomial f. This means that Y is a smooth $\mathbb{Q}$-variety, h is a projective morphism of $\mathbb{Q}$-varieties that is an isomorphism over $\mathbb{A}^n_{\mathbb{Q}} \setminus \mathrm{div}(f)$, and $\mathrm{div}(f \circ h)$ is a strict normal crossings divisor on Y. Such a morphism h always exists, by Hironaka's embedded resolution of singularities in characteristic zero.

(2.2) To every log-resolution h, we associate the following numerical invariants. We write

$$\mathrm{div}(f \circ h) = \sum_{i \in I} N_i E_i$$

where E_i, $i \in I$ are the prime components of the divisor $\mathrm{div}(f \circ h)$, and the N_i are their multiplicities. Since h is an isomorphism over the complement of $\mathrm{div}(f)$, we can write the relative canonical divisor of h as

$$K_{Y/X} = \sum_{i \in I} (\nu_i - 1) E_i.$$

The number ν_i is called the log-discrepancy of X at the divisor E_i; these are fundamental invariants in birational geometry. Roughly speaking, the multiplicities N_i measure the complexity of f and the log-discrepancies ν_i

measure the complexity of the resolution h. For every non-empty subset J of I, we set

$$E_J = \cap_{j\in J} E_j, \quad E_J^o = E_J \setminus (\cup_{i\notin J} E_i).$$

The sets E_J^o form a stratification of $\mathrm{div}(f\circ h)$ into locally closed subsets.

Theorem 2.3 (Igusa 1975)**.** *For every prime number p, the zeta function $Z_{f,p}(s)$ lies in the ring*

$$\mathbb{Q}\left[\frac{1}{p^{as+b}-1}\right]_{a,b\in\mathbb{Z}_{>0}}.$$

If s_0 is a pole of order m of $Z_{f,p}(s)$, then there exists a subset J of I of cardinality m such that $E_J^o(\mathbb{Q}_p)\cap h^{-1}(p\mathbb{Z}_p^n)\neq\emptyset$ and $\Re(s_0) = -\nu_j/N_j$ for every j in J.

Theorem 2.4 (Denef 1991)**.** *If the prime number p is sufficiently large, then*

$$Z_{f,p}(s) = p^{-(n-1)}\sum_{\emptyset\neq J\subset I} \sharp(\overline{E}_J^o(\mathbb{F}_p)\cap \overline{h}^{-1}(0))\prod_{j\in J}\frac{p-1}{p^{N_js+\nu_j}-1}$$

where $\overline{(\cdot)}$ denotes reduction modulo p.

To be precise, Denef's formula is valid when the resolution h has "good reduction modulo p" in a certain technical sense; for our purposes, it suffices to know that this condition is always satisfied for $p\gg 0$.

(2.5) We can draw some immediate consequences from Igusa and Denef's results. Igusa's theorem implies that the real parts of the poles of $Z_{f,p}(s)$ are all contained in the finite set

$$\{-\frac{\nu_i}{N_i}\,|\, i\in I,\, E_i^o(\mathbb{Q}_p)\cap h^{-1}(p\mathbb{Z}_p^n)\neq\emptyset\}.$$

In practice, most of these candidates will not be real parts of actual poles of $Z_{f,p}(s)$. For one thing, the list of candidates strongly depends on the choice of the resolution h, whereas $Z_{f,p}(s)$ only depends on f and p. But even if $n=2$, when there exists a minimal log-resolution h of f, most of the candidates will not appear as real parts of poles of the zeta function. A partial explanation of this phenomenon would be given by the so-called Monodromy Conjecture.

Conjecture (Igusa's Monodromy Conjecture)**.** *If p is sufficiently large and s_0 is a pole of $Z_{f,p}(s)$, then $\Re(s_0)$ is a root of the Bernstein polynomial of f at 0. In particular, $\exp(2\pi i\Re(s_0))$ is a local monodromy eigenvalue of f.*

This conjecture has been solved if $n=2$ by Loeser, and also for some special classes of singularities, but the general case is wide open.

(2.6) Igusa's theorem also implies that the order of a pole s_0 of $Z_{f,p}(s)$ is at most

$$\max\{\sharp J\,|\, J\subset I,\, \Re(s_0) = -\nu_j/N_j \text{ for all } j\in J,\, E_J^o(\mathbb{Q}_p)\cap h^{-1}(p\mathbb{Z}_p^n)\neq\emptyset\}.$$

In particular, it is at most n, since E_J is empty if J has more than n elements ($n+1$ different prime components of a strict normal crossings divisor on a variety of dimension n can never intersect in a point).

(2.7) Finally, from Denef's formula, one can also deduce that the real part of a pole of $Z_{f,p}(s)$ is at most

$$-\min\{\frac{\nu_i}{N_i}\,|\,i\in I,\,E_i\cap h^{-1}(0)\neq\emptyset\}$$

when p is sufficiently large. The number

$$\mathrm{lct}_0(f)=\min\{\frac{\nu_i}{N_i}\,|\,i\in I,\,E_i\cap h^{-1}(0)\neq\emptyset\}$$

is an important invariant in birational geometry, called the log-canonical threshold of f at 0. It is independent of the choice of a log-resolution h. It is used to measure the degree of the singularity of f at 0, and to divide the singularities into different types in the Minimal Model Program.

(2.8) The main subject of these notes is the following conjecture.

Conjecture (Veys 1999)**.** *Assume that p is sufficiently large. If s_0 is a pole of $Z_{f,p}(s)$ of order n, then $\Re(s_0)=-\mathrm{lct}_0(f)$.*

Thus if $Z_{f,p}(s)$ has a pole of the largest possible order (namely, n), then its real part is also as large as possible. Veys's conjecture was originally stated for a different type of zeta function (the so-called topological zeta function), but the proof we will present is valid for the topological and motivic zeta functions, as well.

3. Proof of Veys's conjecture

(3.1) We will deduce Veys's conjecture from a more general result about the geometry of log-resolutions. Let k be a field of characteristic zero, X a smooth k-variety, $f:X\to\mathbb{A}^1_k$ a dominant morphism and $h:Y\to X$ a log-resolution of f as above. We fix a closed point x on X such that $f(x)=0$. The situation we have studied so far corresponds to the case $k=\mathbb{Q}$, $X=\mathbb{A}^n_{\mathbb{Q}}$, $x=0$. We will continue to use the notations N_i, ν_i, E_J etc. The result that we will prove is local on X at the point x; shrinking X around x, we may assume that $E_J\cap h^{-1}(x)\neq\emptyset$ as soon as $E_J\neq\emptyset$, for every non-empty subset J of I. This assumption simplifies some of the notations we will use.

(3.2) For every $i\in I$, we set $\mathrm{wt}_f(E_i)=\nu_i/N_i$ and we call this value the weight of f at E_i. The log-canonical threshold of f at x is then given by

$$\mathrm{lct}_x(f)=\min\{\mathrm{wt}_f(E_i)\,|\,i\in I\}.$$

We write Y_0 for the divisor

$$\mathrm{div}(f\circ h)=\sum_{i\in I}N_iE_i,$$

and we denote by $\Delta(Y_0)$ its dual intersection complex. We introduce a function

$$\mathrm{wt}_f:\Delta(Y_0)\to\mathbb{R}$$

that is completely characterized by the following properties:

- for every $i\in I$, the value of wt_f at the vertex of $\Delta(Y_0)$ corresponding to E_i is given by $\mathrm{wt}_f(E_i)$;

- the function wt_f is affine on every face of $\Delta(Y_0)$.

(3.3) The key observation is that wt_f seems to behave like a "Morse function" on $\Delta(Y_0)$ with unique critical value $\mathrm{lct}_x(f)$, in a sense that is yet to be made precise. We will first look at an analogous setting where the picture is somewhat cleaner. Let Z be a Calabi-Yau variety over $k(\!(t)\!)$ and let ω be a volume form on Z. In a joint work with Mircea Mustaţă, we defined the *weight function*

$$\mathrm{wt}_\omega : Z^{\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$$

on the Berkovich analytification Z^{an} of Z over the non-archimedean field $k(\!(t)\!)$. If $\mathscr{X}$ is a strict normal crossings model for Z over $k[\![t]\!]$, then the dual intersection complex $\Delta(\mathscr{X}_k)$ of its special fiber embeds canonically into Z^{an}. The image of this embedding is called the skeleton $\mathrm{Sk}(\mathscr{X})$ of $\mathscr{X}$. It is a strong deformation retract of Z^{an}. The weight function wt_ω is continuous on $\mathrm{Sk}(\mathscr{X})$ and affine on every face. Moreover, it is strictly decreasing under the projection $Z^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$. In particular, the locus where wt_ω reaches its minimal value $w_{\min}$ on Z^{an} is a union of faces of $\mathrm{Sk}(\mathscr{X})$, which is called the *essential skeleton* $\mathrm{Sk}(Z)$ of Z (it does not depend on ω). The following theorems provide some evidence for the "Morse-like behaviour" of wt_ω.

Theorem A. *If wt_ω is constant on a maximal face σ of $\mathrm{Sk}(\mathscr{X})$, then its value on σ is the minimal value $w_{\min}$ of wt_ω. In other words, σ is contained in the essential skeleton $\mathrm{Sk}(Z)$ of Z.*

Theorem B. *For every $w \in \mathbb{R}$, we denote by $\mathrm{Sk}(\mathscr{X})^{\leq w}$ the subcomplex of $\mathscr{X}$ spanned by the vertices v such that $\mathrm{wt}_\omega(v) \leq w$. There exists a collapse of $\mathrm{Sk}(\mathscr{X})$ onto $\mathrm{Sk}(Z)$ that simultaneously collapses $\mathrm{Sk}(\mathscr{X})^{\leq w}$ for every $w \geq w_{\min}$.*

A collapse is a combinatorial type of strong deformation retract on simplicial complexes. Thus Theorem B implies in particular that the embedding $\mathrm{Sk}(Z) \to \mathrm{Sk}(\mathscr{X})^{\leq w}$ is a homotopy equivalence for every $w \leq w_{\min}$.

(3.4) We have proven similar results in the setting of hypersurface singularities, where log-resolutions play the role of strict normal crossings models. A technical complication is that the components of Y_0 come in two flavours: the exceptional components of h and the components of the strict transform of $\mathrm{div}(f)$. This does not affect the statement of Theorem A (although it makes the proof more difficult) but it forces us to make a case distinction in the formulation of Theorem B. We denote by $\Delta(Y_0)^{=\mathrm{lct}_x(f)}$ the locus where wt_f reaches its minimal value $\mathrm{lct}_x(f)$; this is a union of faces of $\Delta(Y_0)$.

Theorem A'. *If wt_f is constant on a maximal face σ of $\Delta(Y_0)$, then its value on σ is the minimal value $\mathrm{lct}_x(f)$ of wt_f.*

Theorem B'. *Assume that $\mathrm{div}(f)$ is reduced. We denote by $\Delta(Y_0)_{\mathrm{exc}}$ the subcomplex of $\Delta(Y_0)$ spanned by the vertices that correspond to exceptional components of h. For every $w \in \mathbb{R}$, we denote by $\Delta(Y_0)^{\leq w}$ the subcomplex of $\mathscr{X}$ spanned by the vertices v such that $\mathrm{wt}_f(v) \leq w$, and we define $\Delta(Y_0)_{\mathrm{exc}}$ accordingly.*

- *Assume that* $\mathrm{lct}_x(f) = 1$ *(that is,* $(X, \mathrm{div}(f))$ *is log-canonical at* x*). There exists a collapse of* $\Delta(Y_0)$ *onto* $\Delta(Y_0)^{=\mathrm{lct}_x(f)}$ *that simultaneously collapses* $\Delta(Y_0)^{\leq w}$ *onto* $\Delta(Y_0)^{=\mathrm{lct}_x(f)}$ *for every* $w \geq \mathrm{lct}_x(f)$*.*
- *Assume that* $\mathrm{lct}_x(f) \neq 1$*. There exists a collapse of* $\Delta(Y_0)_{\mathrm{exc}}$ *onto* $\Delta(Y_0)^{=\mathrm{lct}_x(f)}$ *that simultaneously collapses* $\Delta(Y_0)^{\leq w}_{\mathrm{exc}}$ *onto* $\Delta(Y_0)^{=\mathrm{lct}_x(f)}$ *for every* $w \geq \mathrm{lct}_x(f)$*.*

Note that in the second case of Theorem B', we have $\mathrm{lct}_x(f) < 1$ so that $\Delta(Y_0)^{=\mathrm{lct}_x(f)}$ is contained in $\Delta(Y_0)_{\mathrm{exc}}$ (components E_i in the strict transform of $\mathrm{div}(f)$ all have $N_i = \nu_i = 1$).

(3.5) Finally, let us explain how Theorem A' implies Veys' conjecture. Let f be a non-constant polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ and let $h : Y \to \mathbb{A}^n_{\mathbb{Q}}$ be a log-resolution for the morphism $f : \mathbb{A}^n_{\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}}$. Suppose that s_0 is a pole of order n of $Z_{f,p}(s)$, for some sufficiently large prime number p. As we have observed above, Denef's formula then implies that there exists a subset J of I of cardinality n such that $E^o_J \cap h^{-1}(0) \neq \emptyset$ and $\Re(s_0) = -\nu_j/N_j$ for every $j \in J$. Each connected component of E^o_J corresponds to a face of $\Delta(Y_0)$ of dimension n on which wt_f is constant with value $-\Re(s_0)$. Since the dimension of $\Delta(Y_0)$ is at most n, such a face is always maximal. Now Theorem A' implies that $\Re(s_0) = -\mathrm{lct}_0(f)$, as predicted by Veys' conjecture.

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