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# A Note on Jacobians, Tutte Polynomials, and Two-Variable Zeta Functions of Graphs

Julien Clancy, Timothy Leake, and Sam Payne
Department of Mathematics, Yale University, New Haven, Connecticut, USA

CONTENTS



We address questions posed by Lorenzini about relations between Jacobians, Tutte polynomials, and the Brill–Noether theory of finite graphs, as encoded in his two-variable zeta functions. In particular, we give examples showing that none of these invariants is determined by the other two.

## 1. INTRODUCTION

The notion of a Riemann–Roch structure on a lattice of rank $n-1$ in $\mathbb{Z}^n$ was recently introduced in [Lorenzini 12], where the author defined a two-variable zeta function associated to each such structure. His construction is inspired by earlier work on two-variable zeta functions for number fields and curves over finite fields [Pellikaan 96, Van der Geer and Schoof 00, Deninger 03, Lagarias and Rains 03] and includes the Riemann–Roch theory for graphs from [Baker and Norine 07] as the special case in which the lattice is the image of the combinatorial Laplacian of a graph on $n$ vertices. This note addresses the relationship between his two-variable zeta function in this special case and more classical invariants of graphs, such as the Tutte polynomial and the Jacobian group, which is the torsion part of the cokernel of the combinatorial Laplacian.

Let $G$ be a finite connected graph without loops or multiple edges, and let $\mathrm{Pic}(G)$ be the cokernel of the combinatorial Laplacian of $G$. In [Baker and Norine 07], a natural degree and rank are assigned to each element of $\mathrm{Pic}(G)$, in close analogy with the degree and rank of divisor classes in the Picard group of an algebraic curve, and Lorenzini's two-variable zeta function, which we denote by $Z_G(t,u)$, encodes the number of divisor classes in $\mathrm{Pic}(G)$ of each degree and rank. More precisely,

$$Z_G(t,u) = \sum_{[D]\in \mathrm{Pic}(G)} \frac{u^{h(D)}-1}{u-1} t^{\deg(D)},$$

where $h(D) = r(D)+1$, one more than the Baker–Norine rank of $D$; it is the analogue of $h^0$ for a divisor on a smooth

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Address correspondence to Sam Payne, Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, CT 06511, USA. Email: sam.payne@yale.edu

algebraic curve. This zeta function is a rational function and can be expressed as

$$\frac{f_G(t,u)}{(1-t)(1-tu)},$$

for some polynomial $f_G$ with integer coefficients. It also satisfies a functional equation

$$Z_G\left(\frac{1}{ut},u\right) = \left(ut^2\right)^{1-g} Z_G(t,u).$$

Furthermore, $f(1,u)$ is the order of the Jacobian group $\mathrm{Jac}(G)$, the group of divisor classes of degree zero [Lorenzini 12, Proposition 3.10], which is equal to the number of spanning trees of $G$.

The Tutte polynomial $T_G(x, y)$ also specializes to the order of the Jacobian group, and graphs with the same Tutte polynomial share many other characteristics. For instance, they have the same number of $k$-colorings for every $k$. For further details on the relationship between Tutte polynomials, chip-firing, and Jacobians of graphs, see [Gabrielov 93a, Gabrielov 93b, Merino López 97, Biggs 99]. It was asked in [Lorenzini 12, p. 20] whether two connected graphs with the same Tutte polynomial must have the same zeta functions or isomorphic Jacobians, and the author observed that the answers are affirmative for trees. Our main results are strong negative answers to both of these questions.

**Theorem 1.1.** *There are pairs of graphs with the same Tutte polynomial and isomorphic Jacobians whose zeta functions are not equal.*

**Theorem 1.2.** *There are pairs of graphs with the same Tutte polynomial and the same zeta function whose Jacobians are not isomorphic.*

We also give a pair of graphs with the same zeta functions and isomorphic Jacobians whose Tutte polynomials are not equal, so no two of these invariants determine the third. In each of these pairs, the two graphs also have the same number of vertices and edges.

**Remark 1.3.** Although any two trees on the same number of vertices have the same Tutte polynomial, they also have trivial Jacobians, and it seems to be difficult to construct large classes of pairs of graphs with the same Tutte polynomial and nontrivial Jacobians. It was conjectured in [Bollobás et al. 00] that the Tutte polynomial (and even its chromatic specialization) is a complete invariant for almost all graphs.

In this project, we pursued two methods for systematically producing graphs with the same Tutte polynomial and nontrivial Jacobians. One method is exhaustive search, which produced most of the examples in Section 3. The other method is Tutte's rotor construction, using rotors of order 3, 4, or 5. A *rotor* is a graph $R$ with an automorphism $\theta$ of the given order $k$ and a vertex $v$ such that $v, \theta v, \ldots, \theta^k v$ are distinct. Tutte's construction takes this rotor together with a map $g$ from the set $\{v, \theta v, \ldots, \theta^k v\}$ to the vertices of another graph $S$ as input. The output is two new graphs, obtained by gluing $R$ to $S$ in different ways, identifying $\theta^i v$ with either $g(\theta^i v)$ or $g(\theta^{k-i} v)$. See [Tutte 74] for further details and a proof that the resulting two graphs have the same Tutte polynomial. Note that the same construction with rotors of higher order generally does not produce graphs with the same Tutte polynomial [Földes 78]. Tutte's rotor construction sometimes produces pairs of graphs whose Jacobians are not isomorphic, as in Example 3.6. Interestingly, however, applying Tutte's construction with his original example of a rotor of order 3 [Tutte 74, Figure 2] has produced pairs of graphs with isomorphic Jacobians in all of our test cases.

**Question 1.4.** Does Tutte's construction with his original example of a rotor of order 3 always produce pairs of graphs with isomorphic Jacobians?

## 2. PRELIMINARIES

Throughout, we consider a finite connected graph $G$, without loops or multiple edges. Let $v_1, \ldots, v_n$ be the vertices of $G$. Recall that a divisor on $G$ is a formal sum $D = a_1 v_1 + \cdots + a_n v_n$ with integer coefficients, and the degree of a divisor is the sum of its coefficients:

$$\deg(D) = a_1 + \cdots + a_n.$$

The combinatorial Laplacian matrix $\Delta(G)$ is the degree matrix minus the adjacency matrix of $G$. Its $i$th diagonal entry is the number of vertices neighboring $v_i$, and its $(i, j)$th off-diagonal entry is $-1$ if $v_i$ is adjacent to $v_j$, and 0 otherwise. The combinatorial Laplacian $\Delta(G)$ determines a map from $\mathbb{Z}^n$ to the group of divisors whose image is a sublattice of rank $n-1$. All divisors in this sublattice have degree zero, so the quotient $\mathrm{Pic}(G)$ is graded by degree. The subgroup consisting of divisor classes of degree zero is called the Jacobian of the graph and is denoted by $\mathrm{Jac}(G)$. Since $G$ is connected, $\mathrm{Jac}(G)$ is the torsion subgroup of the cokernel of $\Delta(G)$. If $G$ has $m$ edges, then the *genus* of $G$ is

$$g = m - n + 1.$$

**Remark 2.1.** This definition of the genus of a graph is the usual one in the literature on tropical geometry and Riemann–Roch theory, and the terminology reflects a close relation to the genus of certain algebraic curves. It should not be confused with the minimal genus of a topological surface in which the graph embeds without crossings, which is also called the genus of the graph in the graph theory literature.

Riemann–Roch theory for graphs, as developed in [Baker and Norine 07], associates an integer rank $r(D)$ to each divisor $D$ on $G$, analogous to the dimension of a complete linear series on an algebraic curve. For the purposes of this note, we follow Lorenzini and work with the invariant $h(D) = r(D) + 1$, which is analogous to the dimension of the space of global sections of a line bundle. It depends only on the class $[D]$ in $\mathrm{Pic}(G)$. Suppose $D$ has degree $d$. If $d$ is negative, then $h(D)$ is zero, and if $d > 2g - 2$, then $h(D) = d - g + 1$.

Brill–Noether theory, for graphs as for algebraic curves, is concerned with the existence and geometry of divisor classes of given degree and rank. For a finite graph, one can simply count these classes, and these counts are encoded in Lorenzini's two-variable zeta function

$$Z_G(t, u) = \sum_{[D] \in \mathrm{Pic}(G)} \frac{u^{h(D)} - 1}{u - 1} t^{\deg(D)}.$$

Note that two graphs have the same zeta function if and only if they have the same number of divisor classes of each degree and rank. In particular, two graphs with the same zeta function have the same number of divisor classes of degree zero, i.e., their Jacobians have the same size. See [Baker 08, Cools et al. 12, Lim et al. 12, Caporaso 12, Len 14] for further details on the Brill–Noether theory of graphs.

## 3. EXAMPLES

Our first example is a pair of graphs with the same Tutte polynomial and isomorphic Jacobians whose zeta functions are different.

**Example 3.1.** Each of the two graphs shown in Figure 1 is a wedge sum of a triangle with the genus-two graph on four vertices; the difference is the vertex of the genus-two graph at which the triangle is attached. Because the Jacobian group of a wedge sum of graphs is the product of the Jacobian groups, and the Tutte polynomial of a wedge sum is the product of the Tutte polynomials, these two graphs have the same Tutte polynomials and isomorphic Jacobians.

One can also compute directly for each graph that the Tutte polynomial is

$$\left(x + x^2 + y\right)\left(x + 2x^2 + x^3 + y + 2xy + y^2\right)$$

and the Jacobian is isomorphic to $\mathbb{Z}/24\mathbb{Z}$. However, $G_2$ has a divisor class of degree 2 and rank 1, represented by twice the rightmost vertex, while $G_1$ has no such divisor class. This can

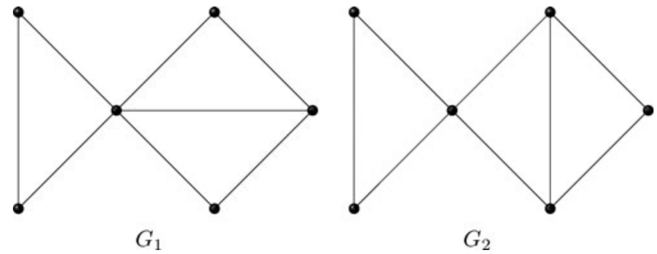


**FIGURE 1.** Example of a pair of graphs with the same Tutte polynomial and isomorphic Jacobians whose zeta functions are different.

be checked directly, by computing the rank of each of the 24 divisor classes of degree 2 on $G_1$ and on $G_2$. Alternatively, one may recall that a graph has a divisor class of degree 2 and rank 1 if and only if it has an involution such that the quotient of the geometric realization by the induced topological involution is a tree [Baker and Norine 09] and observe that $G_2$ has such an involution, given by a vertical reflection in Figure 1, while $G_1$ has no such involution. It follows that the zeta functions of these two graphs are distinct. We find that the zeta functions are

$$Z_{G_1}(t, u) = 1 + 6t + 16t^2 + 6t^3u + t^4u^2 + \frac{24t^3}{(1-t)(1-tu)}$$

and

$$Z_{G_2}(t, u) = 1 + 6t + 16t^2 + t^2u + 6t^3u + t^4u^2 + \frac{24t^3}{(1-t)(1-tu)}.$$

Our next example is a pair of graphs with the same Tutte polynomial and zeta function whose Jacobians are not isomorphic.

**Example 3.2.** Consider the two graphs of genus 4 on eight vertices shown in Figure 2. Each of these graphs has Tutte polynomial

$$\begin{aligned} T(x, y) = {} & x^7 + 4x^6 + x^5y + 9x^5 + 6x^4y + 3x^3y^2 + x^2y^3 \\ & + 13x^4 + 13x^3y + 7x^2y^2 + 3xy^3 + y^4 + 12x^3 \\ & + 15x^2y + 9xy^2 + 3y^3 + 7x^2 + 9xy + 4y^2 \\ & + 2x + 2y \end{aligned}$$

and zeta function

$$Z(t, u) = 1 + 8t + 31t^2 + 77t^3 + 2t^3u + 31t^4u + 8t^5u^2 + t^6u^3 + \frac{125t^4}{(1-t)(1-tu)}.$$

However, their Jacobians are not isomorphic, with

$$\mathrm{Jac}(G_3) \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \quad \text{and} \quad \mathrm{Jac}(G_4) \cong \mathbb{Z}/125\mathbb{Z}.$$

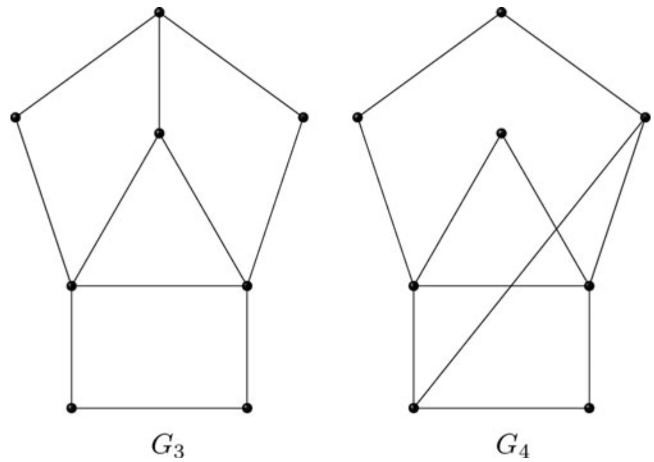


**FIGURE 2.** Example of a pair of graphs with the same Tutte polynomial and zeta function whose Jacobians are not isomorphic.

Examples 3.1 and 3.2 answer Lorenzini's original questions and respectively prove Theorems 1.1 and 1.2.

**Remark 3.3.** The graphs in Example 3.1 have a cut vertex, and those in Example 3.2 can be disconnected by removing two edges, but there are other (more complicated) examples with higher connectivity whose Tutte polynomials, Jacobians, and zeta functions exhibit similar properties. For instance, we found a pair of 3-connected graphs of genus 8 on eight vertices with the same Tutte polynomial and isomorphic Jacobians whose zeta functions are not equal, as well as a pair of 3-connected graphs of genus 10 on nine vertices with the same Tutte polynomial whose Jacobians are not isomorphic.

It is not clear what natural graph-theoretic conditions could imply that two graphs with the same Tutte polynomial must also have the same zeta functions, though we did find experimental evidence suggesting that Tutte's rotor construction with certain rotors might produce pairs of graphs with isomorphic Jacobians. See Remark 1.3 and Question 1.4 above.

**Remark 3.4.** Example 3.2 also gives a negative answer to another question of Lorenzini. It was proved in [Cori and Rossin 00] that planar dual graphs $G$ and $G^*$ have isomorphic Jacobians, and it was asked in [Lorenzini 12, p. 18] whether the existence of this isomorphism follows from the symmetry

$$T_G(x, y) = T_{G^*}(y, x)$$

relating their Tutte polynomials. The graphs $G_3$ and $G_4$ are planar and have the same Tutte polynomial, so we can choose a planar embedding of $G_4$ to obtain a dual graph $G_4^*$. The same symmetry holds, $T_{G_3}(x, y) = T_{G_4^*}(y, x)$, but the Jacobians of $G_3$ and $G_4^*$ are not isomorphic.

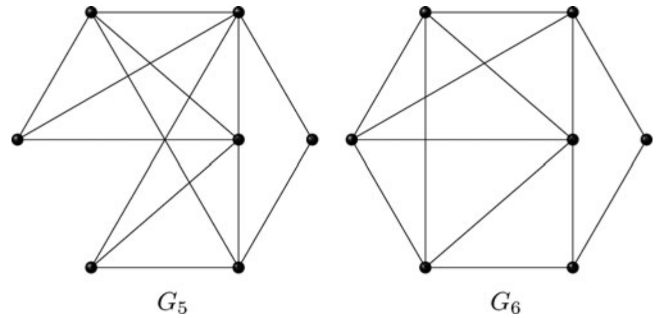


**FIGURE 3.** A pair of graphs with the same zeta function and isomorphic Jacobians whose Tutte polynomials differ.

We also observe that there are pairs of graphs with the same zeta functions and isomorphic Jacobians whose Tutte polynomials differ, as in the following example.

**Example 3.5.** Consider the pair of graphs of genus 7 on seven vertices shown in Figure 3. We find that $G_5$ and $G_6$ both have Jacobians isomorphic to $\mathbb{Z}/545\mathbb{Z}$ and zeta functions equal to

$$\begin{aligned} Z(t, u) = 1 &+ 7t + 27t^2 + 75t^3 + 165t^4 + 299t^5 + 449t^6 \\ &+ 3t^4u + 25t^5u + 105t^6u + 299t^7u + 2t^6u^2 \\ &+ 25t^7u^2 + 165t^8u^2 + 3t^8u^3 + 75t^9u^3 \\ &+ 27t^{10}u^4 + 7t^{11}u^5 + t^{12}u^6 + \frac{545t^7}{(1-t)(1-tu)}. \end{aligned}$$

However, we find that their Tutte polynomials are

$$\begin{aligned} T_{G_5}(x, y) = 8x &+ 26x^2 + 33x^3 + 21x^4 + 7x^5 + x^6 + 8y \\ &+ 41xy + 60x^2y + 34x^3y + 7x^4y + 23y^2 \\ &+ 59xy^2 + 43x^2y^2 + 9x^3y^2 + 29y^3 + 44xy^3 \\ &+ 16x^2y^3 + x^3y^3 + 23y^4 + 22xy^4 + 3x^2y^4 \\ &+ 13y^5 + 7xy^5 + 5y^6 + xy^6 + y^7 \end{aligned}$$

and

$$\begin{aligned} T_{G_6}(x, y) = 10x &+ 27x^2 + 31x^3 + 20x^4 + 7x^5 + x^6 + 10y \\ &+ 45xy + 55x^2y + 32x^3y + 8x^4y + 28y^2 \\ &+ 57xy^2 + 38x^2y^2 + 11x^3y^2 + 34y^3 + 38xy^3 \\ &+ 16x^2y^3 + 2x^3y^3 + 26y^4 + 17xy^4 + 5x^2y^4 \\ &+ 14y^5 + 5xy^5 + x^2y^5 + 5y^6 + xy^6 + y^7. \end{aligned}$$

In the next example, we construct a pair of graphs with the same Tutte polynomial whose Jacobians are not isomorphic using Tutte's rotor construction.

**Example 3.6.** We now apply Tutte's rotor construction to the base graph $G_7$ and rotor $G_8$ to construct two graphs of genus 9 on 11 vertices with the same Tutte polynomial. See Figures 4 and 5.

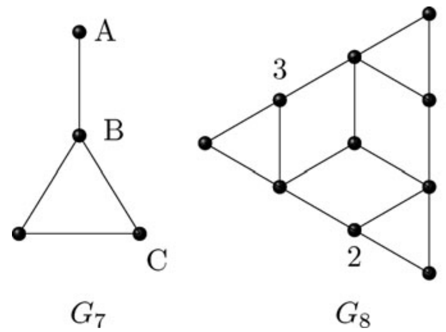


**FIGURE 4.** A base graph ($G_7$) and rotor of order 3 ($G_8$) to which we apply Tutte's rotor construction.

The rotor construction involves gluing the vertices $A$, $B$, and $C$ of $G_7$ to the vertices 1, 2, and 3 of $G_8$ in two different ways. In both cases, we glue $A \mapsto 1$. For $G_9$, we glue $B \mapsto 2$ and $C \mapsto 3$, whereas for $G_{10}$, we glue $B \mapsto 3$ and $C \mapsto 2$, as shown in Figure 5.

By [Tutte 74, 4.1], the graphs $G_9$ and $G_{10}$ have the same Tutte polynomial, which we compute to be

$$\begin{aligned} T(x, y) = {} & 7x + 47x^2 + 139x^3 + 239x^4 + 266x^5 + 202x^6 \\ & + 107x^7 + 39x^8 + 9x^9 + x^{10} + 7y + 72xy \\ & + 270x^2y + 525x^3y + 601x^4y + 426x^5y \\ & + 188x^6y + 49x^7y + 6x^8y + 32y^2 + 213xy^2 \\ & + 553x^2y^2 + 735x^3y^2 + 540x^4y^2 + 217x^5y^2 \\ & + 43x^6y^2 + 3x^7y^2 + 67y^3 + 324xy^3 + 597x^2y^3 \\ & + 525x^3y^3 + 221x^4y^3 + 37x^5y^3 + x^6y^3 + 85y^4 \\ & + 305xy^4 + 392x^2y^4 + 212x^3y^4 + 40x^4y^4 \\ & + 73y^5 + 194xy^5 + 167x^2y^5 + 47x^3y^5 + x^4y^5 \\ & + 45y^6 + y^9 + 86xy^6 + 44x^2y^6 + 4x^3y^6 + 20y^7 \\ & + 25xy^7 + 6x^2y^7 + 6y^8 + 4xy^8. \end{aligned}$$

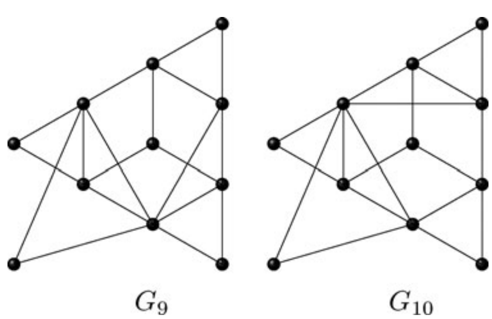


**FIGURE 5.** A pair of graphs with the same Tutte polynomial whose Jacobians are not isomorphic.

However, we find that their Jacobians are not isomorphic, with

$$\begin{aligned} \mathrm{Jac}(G_9) &\cong \mathbb{Z}/9065\mathbb{Z}, \\ \mathrm{Jac}(G_{10}) &\cong \mathbb{Z}/1295\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}. \end{aligned}$$

These two graphs also have distinct zeta functions, with

$$\begin{aligned} Z_{G_9}(t, u) = {} & 1 + 11t + 62t^2 + 241t^3 + 723t^4 + 1757t^5 \\ & + 3529t^6 + 5865t^7 + 8009t^8 + 6t^5u + 86t^6u \\ & + 589t^7u + 2385t^8u + 5865t^9u + 31t^8u^2 \\ & + 598t^9u^2 + 3529t^{10}u^2 + 86t^{10}u^3 \\ & + 1757t^{11}u^3 + 6t^{11}u^4 + 723t^{12}u^5 + 241t^{13}u^5 \\ & + 62t^{14}u^6 + 11t^{15}u^7 + t^{16}u^8 \\ & + \frac{9065t^9}{(1-t)(1-tu)} \end{aligned}$$

and

$$\begin{aligned} Z_{G_{10}}(t, u) = {} & 1 + 11t + 62t^2 + 241t^3 + 723t^4 + 1757t^5 \\ & + 3529t^6 + 5865t^7 + 8009t^8 + 4t^5u + 75t^6u \\ & + 582t^7u + 2369t^8u + 5865t^9u + 37t^8u^2 \\ & + 582t^9u^2 + 3529t^{10}u^2 + 75t^{10}u^3 \\ & + 1757t^{11}u^3 + 4t^{11}u^4 + 723t^{12}u^5 + 241t^{13}u^5 \\ & + 62t^{14}u^6 + 11t^{15}u^7 + t^{16}u^8 \\ & + \frac{9065t^9}{(1-t)(1-tu)}. \end{aligned}$$

**Remark 3.7.** By exhaustive search, we find that there are no pairs of graphs on seven or fewer vertices with the same Tutte polynomial whose Jacobians are not isomorphic. On eight vertices, there are 11 117 isomorphic classes of connected graphs, and we find only two such pairs. These are the pairs in Example 3.2 above, and in Example 3.8 below. On nine vertices, there are 261 080 isomorphism classes of connected graphs, but we find only 122 pairs of graphs with the same Tutte polynomial whose Jacobians are not isomorphic. Some graphs appear in several pairs; these 122 pairs involve 99 different graphs with 33 different Tutte polynomials.

**Example 3.8.** As mentioned above, there are only two pairs of graphs on eight vertices with the same Tutte polynomial whose Jacobians are not isomorphic. One such pair is given in Example 3.2. The other is the pair of graphs of genus 5 shown in Figure 6, which we include for completeness.

Each of these graphs has Tutte polynomial

$$\begin{aligned} T(x, y) = {} & 5x + 18x^2 + 27x^3 + 23x^4 + 13x^5 + 5x^6 + x^7 \\ & + 5y + 25xy + 38x^2y + 25x^3y + 9x^4y + 2x^5y \\ & + 12y^2 + 28xy^2 + 18x^2y^2 + 5x^3y^2 + x^4y^2 \\ & + 11y^3 + 12xy^3 + 3x^2y^3 + 5y^4 + 2xy^4 + y^5 \end{aligned}$$

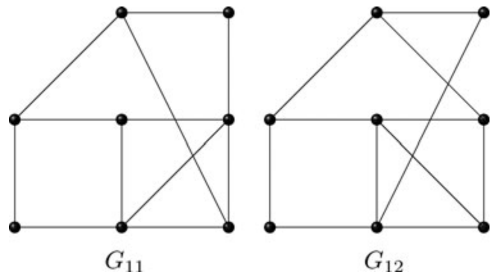


**FIGURE 6.** One of two pairs of graphs on eight vertices with the same Tutte polynomial whose Jacobians are not isomorphic.

and zeta function

$$Z(t,u) = 1 + 8t + 34t^2 + 98t^3 + 202t^4 + 13t^4u + 98t^5u + 34t^6u^2 + 8t^7u^3 + t^8u^4 + \frac{294t^5}{(1-t)(1-tu)}.$$

However, we find that their Jacobians are not isomorphic, with

$$\mathrm{Jac}(G_{11}) \cong \mathbb{Z}/42\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \quad \text{and} \quad \mathrm{Jac}(G_{12}) \cong \mathbb{Z}/294\mathbb{Z}.$$

**Remark 3.9.** Giménez and Merino independently found a pair of graphs with the same Tutte polynomial whose Jacobians are not isomorphic [Giménez and Marino 02]. Their example consists of two planar graphs of genus 10 on 12 vertices whose zeta functions are not equal.

## 4. JACOBIANS OF RANDOM GRAPHS

Jacobians of graphs are frequently cyclic, as has been observed by Lorenzini and others. Perturbing a graph with acyclic Jacobian slightly, by subdividing an edge, tends to produce graphs with cyclic Jacobians. See, for instance, the computation of cyclic Jacobians for modified wheel graphs in [Biggs 07]. This phenomenon was observed again by Robeva in 2008, in computations related to the tropical proof of the Brill–Noether theorem [Cools et al. 12], and we encountered it once more through extensive computations exploring the questions addressed in this paper.

In the course of this investigation, we observed a somewhat more precise structure. The Jacobian group of a graph comes with a canonical duality pairing [Shokrieh 10], and a finite abelian group with duality pairing $(G, \langle\ ,\ \rangle)$ seems to appear with frequency proportional to

$$\frac{1}{|G|\,|\mathrm{Aut}(G, \langle\ ,\ \rangle)|}.$$

Here, $\mathrm{Aut}(G, \langle\ ,\ \rangle)$ denotes the subgroup of the automorphism group of $G$ that preserves the pairing. This experimentally observed variation on the Cohen–Lenstra heuristic [Cohen and Lenstra 84] should explain the prevalence of cyclic Jacobians. We find that the Jacobian of a random graph is cyclic with probability slightly greater than .79. Further details on this heuristic and our experiments with Jacobians of random graphs are presented in [Clancy et al. 14]. Some of our conjectures based on this heuristic are proved in [Wood 14], and connections to symmetric function theory and Hall–Littlewood polynomials are studied in [Fulman 14].

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