TROPICAL BRILL–NOETHER THEORY AND APPLICATIONS I (NOTES FROM THE 2015 SIMONS SYMPOSIUM)

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These notes are from the first of two talks related to ongoing joint work with David Jensen. The content is mostly background material for the second talk, given by Jensen.

1. Classical Brill–Noether theory

Let X be a smooth projective curve of genus g. At a very basic level, we wish to understand the geometry of X by understanding its embeddings in projective space. The space of such embeddings in not compact, so instead we study a compactification $\mathcal{G}^r_d(X)$ parametrizing linear series of degree d and rank r on X. Each such linear series gives a nondegenerate map of degree less than or equal to d to $\mathbb{P}^r$ up to change of coordinates by PGL_{r+1}. The difference between d and the degree of the map is accounted for by the basepoints of the linear series.

The Brill-Noether number

$$\rho(g, r, d) = g - (r + 1)(g - d + r)$$

is a naive dimension estimate for $\mathcal{G}^r_d(X)$. It follows from the Riemann–Roch Theorem that this estimate is correct when $d \geq 2g - 2$. However, for many curves of high genus, including hyperelliptic curves and smooth complete intersections in projective space, this is drastic underestimate when d is small. For instance, there are hyperelliptic curves of every genus even though $\rho(g, 1, 2) = 2 - g$ is negative when g is at least 3. Nevertheless, the estimate is correct on an open dense subset of the space of all curves.

Brill–Noether Theorem. [GH80] *Let X be a general curve of genus g. Then $\mathcal{G}^r_d(X)$ has pure dimension $\rho(g, r, d)$, if this is nonnegative, and is empty otherwise.*

Once we know the dimension of $\mathcal{G}^r_d(X)$, another natural goal is to understand its local structure and any possible singularities. Again, the answer is as simple as possible on an open dense subset of the space of all curves.

Gieseker–Petri Theorem. [Gie82] *Let X be a general curve of genus g. Then $\mathcal{G}^r_d(X)$ is smooth.*

Many other results are known about the geometry of these spaces $\mathcal{G}^r_d(X)$, and we mention just a few. The first three hold for arbitrary curves.

(1) If $\rho(g, r, d) \geq 0$, then $\mathcal{G}^r_d(X)$ is nonempty.
(2) If $\rho(g, r, d) \geq 1$, then $\mathcal{G}^r_d(X)$ is connected.
(3) Every component of $\mathcal{G}^r_d(X)$ has dimension at least $\rho(g, r, d)$.

Date: February 5, 2015.

(4) If X is general and $\rho(g,r,d) \geq 1$ then $\mathcal{G}^r_d(X)$ is irreducible.
(5) If X is general and $0 \geq \rho(g,r,d) \geq g$ then a general linear series on X is complete, i.e. the natural projection from $\mathcal{G}^r_d(X)$ to $\mathrm{Pic}_d(X)$ is generically 1-1 onto its image.

A fundamental difficulty in proving such results about a general curve is that it is extraordinarily difficult to explicitly write down equations for even one sufficiently general curve, when g is large. The original proofs of the Brill–Noether and Gieseker–Petri Theorems were existence proofs via degenerations. Subsequent proofs from the 1980s include one using limit linear series, a systematic approach to the study of degenerations of linear series over one parameter families in which the special fiber is of compact type [EH83], and one that avoids degenerations, instead using vector bundles on $K3$ surfaces and generic smoothness [Laz86]. Here, there is a difficulty in writing down the curves explicitly, since it is extraordinarily difficult to write down explicit equations for $K3$ surfaces of Picard rank 1 with the required degree for each genus. More recently, new proofs of both results have been given via tropical methods, which are most useful for studying degenerations of linear series over one parameter families in which the special fiber is a nodal union of smooth rational curves [CDPR12, JP14].

Some interesting problems about linear series on a general curve remain open, having resisted attempts by all previous methods. Let U be the open subset of the moduli space of curves parametrizing the Brill–Noether–Petri general curves, those curves X for which $\mathcal{G}^r_d(X)$ is smooth of pure dimension $\rho(g,r,d)$ when $\rho(g,r,d)$ is nonnegative and empty otherwise. When $\rho(g,r,d) = 0$, then monodromy acts transitively on the fibers of the universal space of linear series $\mathcal{G}^r_d(U)$. Therefore $\mathcal{G}^r_d(U)$ is irreducible, and it makes sense to talk about a general linear series of degree d and rank r on a general curve X, meaning a linear series corresponding to a point in some open dense subset of $\mathcal{G}^r_d(U)$.

When $\rho(g,r,d) \geq 0$ and $r \geq 3$, a general linear series of degree d and rank r on a general curve of genus g gives an embedding, and the following is an open question. What is the Hilbert function of this embedded curve? A conjectural answer is provided by the Maximal Rank Conjecture, which is attributed to Max Noether.

Maximal Rank Conjecture. *Let $V \subset \mathcal{L}(D_X)$ be a general linear series of rank r and degree d on a general curve X of genus g. Then the multiplication maps*

$$\mu_m : \mathrm{Sym}^m V \to \mathcal{L}(mD_X)$$

have maximal rank for all m.

This conjecture was proved long ago in a number of cases, including when $r \leq 3$ and in the nonspecial case, when $g + r \leq d$ [BE87a, BE87b].

2. Limit linear series

Our primary interest in these talks is to present and explain the tropical approach. For context, we briefly recall the theory of limit linear series for one parameter degenerations to nodal curves of compact type.

Let K be a discretely valued field, such as $\mathbb{C}((t))$, let $R \subset K$ be the valuation ring, and let X be a smooth projective curve of genus g over K. Suppose X has a simple normal crossing semi-stable model whose special fiber is of compact type. In other words, we suppose there is a regular scheme $\mathfrak{X}$ that is proper over $\operatorname{Spec} R$ whose special fiber $\overline{\mathfrak{X}}$ is a reduced nodal curve of compact type. Let $\overline{\mathfrak{X}}_i$ be the irreducible components of $\overline{\mathfrak{X}}$. The compact type condition is equivalent to the condition that $\sum_i g(\overline{\mathfrak{X}}_i) = g$. Another equivalent condition is that the dual graph of $\overline{\mathfrak{X}}$, with one vertex v_i for each component $\overline{\mathfrak{X}}_i$, and one edge from v_i to v_j for each node in $\overline{\mathfrak{X}}_i \cap \overline{\mathfrak{X}}_j$, is a tree. Note that we allow multiple edges in dual graphs, when pairs of components meet at multiple nodes.

Let L be a line bundle of degree d on X. Such a line bundle extends to a line bundle $\mathcal{L}$ on $\mathfrak{X}$, but not uniquely. The basic construction of limit linear series involves choosing distinguished extensions of L, as follows. For each irreducible component $\overline{\mathfrak{X}}_i$ of $\overline{\mathfrak{X}}$, there is a unique extension $\mathcal{L}_i$ such that

$$\deg(\mathcal{L}_i|_{\overline{\mathfrak{X}}_j}) = d \cdot \delta_{ij}.$$

Then, given a linear series $W \subset H^0(X, L)$, we consider the collection of linear series $\{W_i \subset H^0(\overline{\mathfrak{X}}_i, \mathcal{L}_i)\}$ consisting of sections of these distinguished sections that are limits of sections in W. These collections of linear series $\{W_i\} \in \prod_i \mathcal{G}^r_d(\overline{\mathfrak{X}}_i)$ satisfy a compatibility condition, which may be expressed simply in terms of vanishing sequences at the nodes of $\overline{\mathfrak{X}}$. The theory of limit linear series studies all such collections of linear series, satisfying this compatibility condition. One consequence of this theory is that if $\overline{\mathfrak{X}}$ is a chain of g curves of genus 1, and if the difference between the two nodes on each of the $g-2$ curves in the interior of the chain do not differ by a torsion point of order less than or equal to $2g-2$ in the Jacobian, then $\overline{\mathfrak{X}}$ is not in the closure of complement of U in the moduli space of stable curves.

3. The tropical approach

Consider X, $\mathfrak{X}$, and L as above, but we no longer assume that $\overline{\mathfrak{X}}$ is of compact type. The tropical approach can be adapted to work much more generally, but for simplicity we assume that each component $\overline{\mathfrak{X}}_i$ is isomorphic to $\mathbb{P}^1$. Let G be the dual graph of $\overline{\mathfrak{X}}$. The condition that each $\overline{\mathfrak{X}}_i$ is a smooth rational curve is equivalent to the condition that $h^1(G) = g$. Here, the first Betti number $h^1(G)$ is the number of vertices in G, minus the number of edges, plus one.

Since we are no longer in the case where the dual graph G is a tree, there may or may not exist an extension $\mathcal{L}_i$ of L such that

$$\deg(\mathcal{L}_i|_{\overline{\mathfrak{X}}_j}) = d \cdot \delta_{ij}.$$

Lacking these distinguished extensions, we instead study the obstruction to finding such extensions, which can be measured by the component group of the Nëron model of the Jacobian variety $\operatorname{Jac}(X)$. Most importantly for our purposes, this component group depends only on the dual graph G and can be described as follows.

Let $\Delta(D)$ be the combinatorial Laplacian matrix, which is the degree matrix minus the adjacency matrix. Its rows and columns are indexed by the vertices v_i of G (or, equivalently, by the components $\overline{\mathfrak{X}}_i$ of $\overline{\mathfrak{X}}$). The ith diagonal entry is the number of edges incident to v_i (equivalently, the number of nodes on $\overline{\mathfrak{X}}_i$) and the (i, j) off-diagonal entry is minus the number of edges joining v_i to v_j. One can think of $\Delta(G)$ as an operator

$$\Delta(G) : \mathbb{Z}^{\mathrm{vert}(G)} \to \mathbb{Z}^{\mathrm{vert}(G)}.$$

The rows and columns of $\Delta(G)$ all sum to zero, so the matrix is singular. In fact, it has corank 1. Then the component group of the Néron model of $\mathrm{Jac}(X)$ is isomorphic to the torsion subgroup of the cokernel of $\Delta(G)$. This finite abelian group has size equal to the number of spanning trees of G. It has many names, and has been called the sandpile group of G and the critical group of G, among others. We follow the tradition in the tropical geometry literature, calling it the Jacobian of G and denoting it by $\mathrm{Jac}(G)$.

We think of the target of $\Delta(G)$, consisting of formal $\mathbb{Z}$-linear combinations of vertices, as divisors on the graph, and then the image of $\Delta(G)$ is the subgroup of principal divisors. A divisor $a_0v_0 + \cdots + a_sv_s$ has degree $a_0 + \cdots + a_s$, and the image of $\Delta(G)$ is contained in the subgroup of divisors of degree zero. In the terms of this analogy, the group $\mathrm{Jac}(G)$ is the divisors of degree zero modulo the principal divisors. The group $\mathrm{Jac}(G)$ has a distinguished generating set, consisting of the classes of divisors $v_i - v_0$, which is analogous to the image of the Abel–Jacobi map. One key technical fact is that $\mathrm{Jac}(G)$, with this generating set, has diameter equal to g—the first Betti number of the graph G, which is the same as the genus of the smooth projective curve X.

In order to relate $\mathrm{Jac}(G)$ to Brill–Noether theory, one needs a suitable analogue of the rank of a divisor on an algebraic curve. There are now several such analogues in the literature, and most of them satisfy analogues of the Riemann–Roch Theorem and are related to ranks of divisors on smooth projective curves through a specialization lemma. We use the original combinatorial rank function defined by Baker and Norine, which has the advantage of being efficiently computable via Dhar's burning algorithm.

Just as for divisors on algebraic curves, we say that a divisor $D = a_0v_0 + \vdots + a_sv_s$ on G is effective if all of the coefficients a_i are nonnegative, and that two divisors are equivalent if their difference in principal.

Definition 3.1. *The rank of a divisor D on G is the largest integer r such that $D - E$ is equivalent to an effective divisor for every effective divisor E of degree r.*

Note that the analogous property is an equivalent characterization of the rank of a divisor on an algebraic curve.

These ranks of divisors on graphs are related to ranks of divisors on curves through Baker's specialization lemma [**?**], as follows. Let $\mathcal{L}$ be any extension of L to $\mathfrak{X}$, and let $a_i = \deg(\mathcal{L}|_{\mathfrak{X}_i})$. Then the rank of $a_0v_0 + \cdots + a_sv_s$ is at least the rank of L.

In order to give a tropical proof of the Brill–Noether theorem, one then needs a suitable class of graphs on which all of the divisor classes of degree d and rank at least r can be explicitly understood. Only one such class of graphs is known at this time—the chain of loops with generic edge lengths. The computations needed to prove the Brill–Noether theorem are carried out on these chains of loops in [CDPR12].

To go deeper into this subject (e.g. to prove Gieseker–Petri) one needs the tropical Riemann-Roch Theorem, also due to Baker and Norine, which says that $r(D) - R(K_G - D) = \deg(D) + 1 - g$, for all divisors D on G. To go even further (e.g. to prove cases of the Maximal Rank Conjecture) one also needs a suitable lifiting theorem. For chains of loops, this is proved in [?], using Rabinoff's lifting theorem for complete intersections of analytic hypersurfaces [?].

References

[Bak08] M. Baker, *Specialization of linear systems from curves to graphs*, Algebra Number Theory **2** (2008), no. 6, 613–653.

[BE87a] E. Ballico and P. Ellia, *Beyond the maximal rank conjecture for curves in* $\mathbf{P}^3$, Space curves (Rocca di Papa, 1985), Lecture Notes in Math., vol. 1266, Springer, Berlin, 1987, pp. 1–23.

[BE87b] ———, *The maximal rank conjecture for nonspecial curves in* $\mathbf{P}^n$, Math. Z. **196** (1987), no. 3, 355–367.

[CDPR12] F. Cools, J. Draisma, S. Payne, and E. Robeva, *A tropical proof of the Brill-Noether theorem*, Adv. Math. **230** (2012), no. 2, 759–776.

[CJP14] D. Cartwright, D. Jensen, and S. Payne, *Lifting divisors on a generic chain of loops*, To appear in Canad. Math. Bull. arXiv:1404.4001, 2014.

[EH83] D. Eisenbud and J. Harris, *On the Brill-Noether theorem*, Algebraic geometry—open problems (Ravello, 1982), Lecture Notes in Math., vol. 997, Springer, Berlin, 1983, pp. 131–137.

[GH80] P. Griffiths and J. Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. **47** (1980), no. 1, 233–272.

[Gie82] D. Gieseker, *Stable curves and special divisors: Petri's conjecture*, Invent. Math. **66** (1982), no. 2, 251–275.

[JP14] D. Jensen and S. Payne, *Tropical multiplication maps and the Gieseker-Petri Theorem*, preprint, arXiv:1401.2584, 2014.

[Laz86] R. Lazarsfeld, *Brill-Noether-Petri without degenerations*, J. Differential Geom. **23** (1986), no. 3, 299–307.

[Rab12] J. Rabinoff, *Tropical analytic geometry, Newton polygons, and tropical intersections*, Adv. Math. **229** (2012), no. 6, 3192–3255.