Polynomial Point Counts and Odd Cohomology Vanishing on Moduli Spaces of Stable Curves

forthcoming joint work w/ J. Bergström and C. Faber

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All cohomology w/ $Q$-coeffs.

Inspiration & Motivating Question from:

- “Calculating Cohomology Groups of Moduli Spaces of Curves via Algebraic Geometry”

**Theorem (AC98)** $H^k(\bar{M_g},n) = 0$ for $k \in \{1, 3, 5\}$

**Example:** $H^2(\bar{\mathcal{M}_{1,11}}) \cong Q^2$

**Q:** (AC98) What about $k \in \{7, 9\}$?
Odd cohomology of Open Moduli Spaces:

**Thm (Madsen Weiss 2007)** (also Mumford, Harer, Lodijkgo, Tillmann, ...)

For \( * < 2g/3 \), \( H^*(Ugim) \cong \mathbb{Q}[y_1, \ldots, y_9, k_1, k_2, \ldots] \)

**Thm (Harer Zagier 1986)**

\((-1)^{g+1} X(Ug) \sim g^{2g}\)

**Q: (HZ 86)** Where is the odd cohomology?

**Thm (Tommasi 2005)**

\( H^5(U4) \cong \mathbb{Q} \) (wt 6)

**Thm (Chen Galatius P- 2021)**

\( H^5(U6) \neq 0; \ H^{23}(U_8) \neq 0; \ H^{27}(U_{10}) \neq 0 \) (top weight)
Relation to odd cohomology for $\overline{\text{M}}_{g,n}$: (Excision)

\[ \cdots \to H^c_c(M_{g,n}) \to H^c(\overline{\text{M}}_{g,n}) \to H^c(\mathcal{D}M_{g,n}) \to \cdots \]

Thm (AC98) Suppose $H^c_c(M_{g,n}) = 0$.

Then $H^c(\overline{\text{M}}_{g,n}) \to H^c(\mathcal{D}M_{g,n})$ is inj.

Proof: a little bit of mixed Hodge theory.

Corollary: Suppose $k$ is odd and $H^i_\text{dR}(\overline{\text{M}}_{g',n'}) = 0$ for odd $i \leq k$, and $(g',n') <_{\text{lex}} (g,n)$. Then

$H^c_c(M_{g,n}) = 0 \Rightarrow H^c(\overline{\text{M}}_{g,n}) = 0$. 
**Thm** (Harer 1984) + (Church Faulkner Putman 2012/Naito Sakasai Suzuki 2013)

\[
H_c^k(M_{g,n}) = 0 \quad \text{for} \quad \begin{cases} 
\mu = 0, 1 & k < 2g \\
n > 1 & k < 2g - 2 + n
\end{cases}
\]

**Cor** (AC98)

Suppose $K$ is odd and $H^1(M_{g,n}) = 0$ for odd $g \neq K$ and all $g,n$. Then to prove $H_c^k(M_{g,n}) = 0$ for all $g,n$, it suffices to check finitely many cases.

**Thm** (Bergström Faber P 2021)

$H^7(M_{g,n}) = 0$ and $H^9(M_{g,n}) = 0$ for all $g,n$. 
Required cases:

\[ g = 0 \quad (\text{Keal 1992}) \]
\[ g = 1 \quad (\text{Getzler 1997, Petersen 2014}) \]
\[ g = 2 \quad (\text{Bergström 2009, Petersen-Tommasi 2014}) \]
\[ g = 3 \quad (\text{Bergström 2008}) \]

\[ \Rightarrow \quad \text{Thm for } H^7(\bar{\mu}_{g,n}) \]

For \( H^9(\bar{\mu}_{g,n}) \),

\[ g = 4, n = 0 \quad (\text{Tommasi 05}) \]

Remains to check:

\[ H^9(\bar{\mu}_{4,n}) = 0 \quad \text{for } n \in \{1, 2, 3\} \]
Polynomial Point Counts

**Thm (vanden Bogaart Edixhoven 2005)**

Let \( X \) be a DM stack that is smooth and proper over \( \mathbb{Z}[\frac{1}{7}] \).

Then \( H^*(X) \) is pure Hodge-Tate if and only if \( \#X(\mathbb{F}_7) \)

is a polynomial in \( q \) (for \( q = p^r \), \( p \) the Spec \( \mathbb{Z}[\frac{1}{7}] \)).

(Proof: Behrend's trace formula and the Weil conjectures)

**Thm (Bergström Faber P 2021)**

\( \# \overline{M}_{4,n}(\mathbb{F}_7) \) is a polynomial in \( q \), for \( n \leq 3 \)

Also: Computed the polynomials and \( S_n \in \mathbb{C} H^*(\overline{M}_{4,n}) \)

(modulo a few details...)
Polynomials: (still conjectural for $n=2,3$; confirmed at $g=1,2,3$).

- $\overline{M}_{4,1}(F_3) = 2^{10} + 6g^9 + 30g^8 + 93g^7 + 191g^6 + 240g^5 + \cdots$

- $\overline{M}_{4,2}(F_3) = 3^{11} + 11g^{10} + 76g^9 + 319g^8 + 838g^7 + 1362g^6 + \cdots$

- $\overline{M}_{4,3}(F_3) = 3^{12} + 21g^{11} + 207g^{10} + 1168g^9 + 3977g^8 + 8296g^7 + 10605g^6 + \cdots$

Polynomials in Open Moduli

- $M_{4,1}(F_3) = 3^{10} + 2g^9 + 2g^8 - 2g^7 - 2g^6 - 3^2$

- $M_{4,2}(F_3) = 3^{11} + 3g^{10} + 4g^9 - 2g^8 - 4g^7 - 2g^6 - 2g^2$

- $M_{4,3}(F_3) = 3^{12} + 4g^{11} + 7g^{10} - 4g^9 - 13g^8 + 4g^7 - 3g^6 - 11g^3 + 2g^2 + 2g - 1$

eg. $M_{4,3}(F_3) = 1497092$ (confirmed)
Connections to Langlands Program

Simple pure motives in smooth proper DM stacks over $\mathbb{Z}$

\[ \cong \]
polarized algebraic cuspidal automorphic representations of conductor 1.

**Thm (Chenevier-Lannes 2019)**

Classification on automorphic side for weights \(< 23:

\[ 1; \Delta_{11}; \Delta_{15}; \Delta_{17}; \Delta_{19}; \Delta_{19,7}; \Delta_{21}; \Delta_{21,5}; \Delta_{21,9}; \Delta_{21,13}; \text{Sym}^2 \Delta_{11} \]

**Predictions (Assuming Langlands Correspondence)**

- No curves smooth and proper over $\mathbb{Z}$
  (or principally polarized abelian varieties - wt 1) \(\text{(Thm: Fontaine 1985)}\)
- If \(X\) is smooth and proper over $\mathbb{Z}$ then \(H^k(X) = 0\) for odd \(k < 11\).
• If $X$ is smooth and proper over $\mathbb{Z}$ and $X_{\mathbb{C}}$ is unirational, then $H^m(X) = 0$.

• If $\dim X \leq 10 \leq \dim X \leq 12$ and $X_{\mathbb{P}}$ is unirational, then $\# X(\mathbb{F}_q)$ is a polynomial in $q$.

**NB:** $\overline{U}_{4,n}$ is unirational for $n < 15$. (polynomial point count for $n \leq 3$) conditional on Langlands

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**A sample Calculation for $\overline{M}_4$, $M_4$**

$\# \overline{M}_4(\mathbb{F}_q) = 4q^9 + 28q^8 + 94q^7 + 192q^6 + 240q^5 + 191q^4 + \ldots$

$\ldots + 93q^3 + 31q^2 + 6q + 1$

$M_4 = H_4 \cup P_4 \cup U_4$

hyperelliptic, Petri, special
\[ \# H_4(\mathbb{F}_3) = q^7 \]

Claim: \[ \# P_4(\mathbb{F}_3) = q^8 + O(q^6) \]

Sketch: Every curve in $P_4$ is a weighted degree 6 curve on the quadric cone $\mathbb{P}(1,1,2)$.

Consider $A^5 = \frac{1}{2}$ curves in $\mathbb{P}(6)$ not passing through the curve pt

- Count smooth curves in $A^5$ and divide by $\# Aut(\mathbb{P}(1,1,2))(\mathbb{F}_3) = q^7 - q^6 - q^5 + q^4$.
- Start w/ $q^5$ curves
- Subtract terms for curves singular at one point
  \[- (q^2 + q) \cdot q^{12} \]
- Running total: $q^{15} - q^{14} - q^{13}$.
• **Sieve**: Add terms for curves singular at two $\mathbb{F}_q$-points, and subtract terms for curves singular at a conjugate pair of points over $\mathbb{F}_{q^2}$.

  (Note: the points must be non-collinear!)

  \[
  \frac{1}{2} (q^2+1) q^2 \cdot q^9 - \frac{1}{2} (q^4 - q) - (q^3 - q) \cdot q^9
  \]

• **Running total**: $q^{15} - q^{14} - q^{13} + q^{12}$

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**Sieve via Hasse-Weil Zeta functions** (Wennink 2020)

**Thm** (Vakil Wood 2015)

\[
\frac{1}{Z(Y; t)} = \sum \left( \sum (-1)^{d(x)} \cdot \# Y(\kappa) \right) \cdot t^d
\]

\{
\text{Frob stable subsets of } Y(\mathbb{F}_q) \}

w/ orbit type $\lambda$
Write: \[ \frac{1}{Z(Y, t)} = \sum \frac{S_d(x)}{d} t^d. \]

Examples:

\[ Y = \mathbb{P}(1,1,2)^{sm} \]

\[ S_1(Y) = -8^2 - 2 \]

\[ S_2(Y) = 8^3 \]

\[ S_d(Y) = 0, \quad d \geq 3 \]

• Terms for two singular points, reprise:

\[ S_2(Y) \cdot 8^9 + S_2(A') \cdot _- = 8^{12}. \]

• Terms for three singular points

\[ S_3(Y) \cdot _7^6 + S_2(A') \cdot _- + S_3(A') \cdot _- = 0. \]
• Conclude: \( * P_4(F_{q^2}) = q^8 + O(q^n) \)

• Similarly: \( * U_4(F_{q^2}) = q^9 - q^6 + O(q^n) \)

... Add \( * H(F_{q^2}) + * \mathbb{H}_4(F_{q^2}) \) and apply Poincaré duality to get \( * \mathbb{H}_4(F_{q^2}) \) ...