Polynomial Point Counts and Odd Cohomology Vanishing on Moduli Spaces of Stable Curves

forthcoming joint work w/ J. Bergström and C. Faber

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All cohomology w/ $\mathbb{Q}$-coeffs.

Inspiration + Motivating Question from:

- "Calculating Cohomology Groups of Moduli Spaces of Curves via Algebraic Geometry"

  Arbarello and Cornalba, Publ Math IHES 1998.

<u>Thm</u> (AC98) $H^k(\overline{M}_{g,n}) = 0$ for $k \in \{1,3,5\}$

<u>Example</u>: $H^{11}(\overline{M}_{1,11}) \cong \mathbb{Q}^2$

$$\boxed{\underline{Q}: \text{ (AC98) What about } k \in \{7,9\}?}$$

Odd cohomology of Open Moduli Spaces:

Thm (Madsen Weiss 2007) (also: Mumford, Harer, Looijenga, Tillmann, ...)

For $* < 2g/3$, $H^*(\mathcal{M}_{g,n}) \cong \mathbb{Q}[\psi_1, \dots, \psi_n, \kappa_1, \kappa_2, \dots]$

Thm (Harer Zagier 1986)

$$(-1)^{g+1} \chi(\mathcal{M}_g) \sim g^{2g}$$

Q: (HZ 86) Where is the odd cohomology?

Thm (Tommasi 2005)

$$H^5(\mathcal{M}_4) \cong \mathbb{Q} \quad (\text{wt } 6)$$

Thm (Chan Galatius P- 2021)

$$H^{15}(\mathcal{M}_6) \neq 0; \; H^{23}(\mathcal{M}_8) \neq 0; \; H^{27}(\mathcal{M}_{10}) \neq 0$$

(top weight)

Relation to odd cohomology for $\overline{M}_{g,n}$: (Excision)

$$\cdots \to H^k_c(\mathcal{M}_{g,n}) \to H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\mathcal{M}_{g,n}) \to \cdots$$

<u>Thm</u> (AC98) Suppose $H^k_c(\mathcal{M}_{g,n}) = 0$.

Then $H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\widetilde{\mathcal{M}}_{g,n})$ is inj.

normalization of $\partial\mathcal{M}_{g,n}$.

<u>Proof</u>: a little bit of mixed Hodge theory

<u>Corollary</u>: Suppose k is odd and $H^j(\overline{\mathcal{M}}_{g',n'}) = 0$ for odd $j \leq k$ and $(g',n') <_{lex} (g,n)$. Then

$$H^k_c(\mathcal{M}_{g,n}) = 0 \Rightarrow H^k(\overline{\mathcal{M}}_{g,n}) = 0.$$

Thm (Harer 1984)+(Church Farb Putman 2012 / Morita Sakasai Suzuki 2013)

vcd +ε

$$H_c^k(\mathcal{M}_{g,n})=0 \quad \text{for} \quad \begin{cases} n=0,1 & k<2g \\ n>1 & k<2g-2+n \end{cases}$$

Cor (AC98)

Suppose k is odd and $H^j(\overline{\mathcal{M}}_{g,n})=0$ for odd $j<k$ and all g,n. Then to prove $H^k(\overline{\mathcal{M}}_{g,n})=0$ for all g,n, it suffices to check finitely many cases.

Thm (Bergström Faber P 2021)

$H^7(\overline{\mathcal{M}}_{g,n})=0$ and $H^9(\overline{\mathcal{M}}_{g,n})=0$ for all g,n.

Required cases: $g=0$ (Keel 1992)

$g=1$ (Getzler 1997, Petersen 2014)

$g=2$ (Bergström 2009, Petersen-Tommasi 2014).

$g=3$ (Bergström 2008)

$\Rightarrow$ Thm for $H^7(\overline{\mathcal{M}}_{g,n})$

For $H^9(\overline{\mathcal{M}}_{g,n})$,

$g=4, n=0$ (Tommasi 05)

Remains to check:

$$H^9(\overline{\mathcal{M}}_{4,n})=0 \text{ for } n\in\{1,2,3\}.$$

Polynomial Point Counts

Thm (van den Bogaart Edixhoven 2005)

Let X be a DM stack that is smooth and proper over $\mathbb{Z}[\frac{1}{n}]$

Then $H^*(X)$ is pure Hodge-Tate if and only if $\# X(\mathbb{F}_q)$ is a polynomial in q (for $q = p^n$, $(p) \in \operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$).

(Proof: Behrend's trace formula and the Weil conjectures)

Thm (Bergström Faber P 2021)

$\# \overline{\mathcal{M}}_{4,n}(\mathbb{F}_q)$ is a polynomial in q, for $n \leq 3$

Also: Computed the polynomials and $S_n \circlearrowright H^*(\overline{\mathcal{M}}_{4,n})$

(modulo a few details ...)

Polynomials: (still conjectural for $n=2,3$; confirmed at $g=1,2,3$).

Gorsky Xarles/Faber

$$\# \overline{\mathcal{M}}_{4,1}(\mathbb{F}_q) = q^{10} + 6q^9 + 30q^8 + 93q^7 + 191q^6 + 240q^5 + \cdots$$

tautological

$$\# \overline{\mathcal{M}}_{4,2}(\mathbb{F}_q) = q^{11} + 11q^{10} + 76q^9 + 319q^8 + 838q^7 + 1362q^6 + \cdots$$

$$\# \overline{\mathcal{M}}_{4,3}(\mathbb{F}_q) = q^{12} + 21q^{11} + 207q^{10} + 1168q^9 + 3977q^8 + 8296q^7 + 10605q + \cdots$$

Polynomials for Open Moduli

$$\# \mathcal{M}_{4,1}(\mathbb{F}_q) = q^{10} + 2q^9 + 2q^8 - q^7 - q^6 - q^2$$

$$\# \mathcal{M}_{4,2}(\mathbb{F}_q) = q^{11} + 3q^{10} + 4q^9 - 2q^8 - 4q^7 - 2q^3 - 2q^2$$

$$\# \mathcal{M}_{4,3}(\mathbb{F}_q) = q^{12} + 4q^{11} + 7q^{10} - 4q^9 - 13q^8 + 4q^7 - q^6 - 11q^3 + 2q^2 + 2q - 1$$

eg. $\# \mathcal{M}_{4,3}(\mathbb{F}_5) = 1\,497\,092$ (confirmed)

Connections to Langlands Program

simple pure motives in smooth proper DM stacks over $\mathbb{Z}$ $\xleftrightarrow{\text{Conj}}$ polarized algebraic cuspidal automorphic representations of conductor 1.

Thm (Chenevier Lannes 2019)

Classification on automorphic side for weights < 23:

$1;\ \Delta_{11};\ \Delta_{15};\ \Delta_{17};\ \Delta_{19};\ \Delta_{19,7};\ \Delta_{21};\ \Delta_{21,5};\ \Delta_{21,9};\ \Delta_{21,13};\ \mathrm{Sym}^2\Delta_{11}$

Predictions (Assuming Langlands Correspondence)

- No curves smooth and proper over $\mathbb{Z}$ (Thm: Fontaine 1985)
  (or principally polarized abelian varieties – wt 1)
- If X is smooth and proper over $\mathbb{Z}$ then $H^k(X) = 0$ for odd $k < 11$.

- If X is smooth and proper over $\mathbb{Z}$ and $X_{\mathbb{C}}$ is unirational then $H^{11}(X) = 0$.
- If $\dim X \leq 10$ or $\dim X \leq 12$ and $X_{\mathbb{C}}$ is unirational then $\# X(\mathbb{F}_q)$ is a polynomial in q.

<u>NB</u>: $\overline{\mathcal{M}}_{4,n}$ is unirational for $n \leq 15$. ($\Rightarrow$ polynomial point count for $n \leq 3$) conditional on Langlands

A sample Calculation for $\overline{\mathcal{M}}_4$, $\mathcal{M}_4$

$$\# \partial \mathcal{M}_4(\mathbb{F}_q) = 4q^9 + 28q^8 + 94q^7 + 192q^6 + 240q^5 + 191q^4 + \cdots$$
$$\cdots + 93q^3 + 31q^2 + 6q + 1$$

$$\mathcal{M}_4 = \underset{\text{hyperelliptic}}{H_4} \sqcup \underset{\text{Petri special}}{P_4} \sqcup U_4$$

$\# H_4(\mathbb{F}_q) = q^7$

<u>Claim</u>: $\# P_4(\mathbb{F}_q) = q^8 + O(q^4)$

<u>Sketch</u>: Every curve in P_4 is a weighted degree 6 curve on the quadric cone $\mathbb{P}(1,1,2)$.

Consider $A^{15} = \{$ curves in $|\mathcal{O}(6)|$ not passing through the cone pt $\}$

- Count smooth curves in A^{15} and divide by $\# \mathrm{Aut}(\mathbb{P}(1,1,2))(\mathbb{F}_q)$
$= q^7 - q^6 - q^5 + q^4$.

- Start w/ q^{15} curves
- Subtract terms for curves singular at one point

$$-(q^2+q)\cdot q^{12}$$

- Running total: $q^{15} - q^{14} - q^{13}$.

- Sieve: Add terms for curves singular at two $\mathbb{F}_q$-points, and subtract terms for curves singular at a conjugate pair of points over $\mathbb{F}_{q^2}$.

(Note: the points must be non-collinear!)

$$\frac{1}{2}(q^2+q)q^2 \cdot q^9 \qquad - \frac{1}{2}\left((q^4-q)-(q^3-q)\right)\cdot q^9$$

- Running total: $q^{15} - q^{14} - q^{13} + q^{12}$

Sieve via Hasse-Weil Zeta functions (Wennink 2020)

Thm (Vakil Wood 2015)

$$\frac{1}{Z(Y;t)} = \sum_d \left(\sum_{\lambda \vdash d} (-1)^{\ell(\lambda)} \cdot \# Y(\lambda) \right) \cdot t^d$$

$\{$ Frob stable subsets of $Y(\overline{\mathbb{F}}_q)$ w/ orbit type λ $\}$

Write: $\dfrac{1}{Z(Y;t)} = \sum_d s_d(X) t^d$.

Examples: $Y = \mathbb{P}(1,1,2)^{sm}$

$s_1(Y) = -q^2 - q$ $\qquad$ $s_d(\mathbb{A}^n) = 0$, $d \geq 2$.

$s_2(Y) = q^3$

$s_d(Y) = 0$, $d \geq 3$

- Terms for two singular points, reprise:

$$s_2(Y) \cdot q^9 + s_2(\mathbb{A}^1) \cdot \_\_ = q^{12}.$$

- Terms for three singular points

$$s_3(Y) \cdot q^6 + s_2(\mathbb{A}^1) \cdot \_\_ + s_3(\mathbb{A}^1) \cdot \_\_ = 0$$

- Conclude $\# P_4(\mathbb{F}_q) = q^8 + O(q^4)$
- <u>Similarly</u>: $\# U_4(\mathbb{F}_q) = q^9 - q^6 + O(q^4)$

... Add $\# H(\mathbb{F}_q) + \# \partial M_4(\mathbb{F}_q)$ and apply Poincaré duality to get $\# \overline{M}_4(\mathbb{F}_q)$ ...