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On the structure of non-archimedean analytic curves

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Abstract. Let K be an algebraically closed, complete non-Archimedean field and let X be a smooth K-curve. In this paper we elaborate on several aspects of the structure of the Berkovich analytic space X^{an}. We define semistable vertex sets of X^{an} and their associated skeleta, which are essentially finite metric graphs embedded in X^{an}. We prove a folklore theorem which states that semistable vertex sets of X are in natural bijective correspondence with semistable models of X, thus showing that our notion of skeleton coincides with the standard definition of Berkovich [**4**]. We use the skeletal theory to define a canonical metric on $\mathbb{H}(X^{\mathrm{an}}) := X^{\mathrm{an}} \setminus X(K)$, and we give a proof of Thuillier's non-Archimedean Poincaré-Lelong formula in this language using results of Bosch and Lütkebohmert.

1. Introduction

Throughout this paper we let K denote an algebraically closed field which is complete with respect to a nontrivial, non-Archimedean valuation $\mathrm{val}\colon K \to \mathbb{R} \cup \{\infty\}$. Let R be the valuation ring of K, $\mathfrak{m}_R$ its maximal ideal, and k its residue field. In this situation, R is not Noetherian and k is algebraically closed. We let $|\cdot| = \exp\bigl(-\mathrm{val}(\cdot)\bigr)$ be an associated absolute value and let $G = \mathrm{val}(K^\times) \subset \mathbb{R}$ be the value group.

1.1. Let X be a smooth, proper, connected algebraic K-curve and let X^{an} be its analytification in the sense of Berkovich [**4**]. The purpose of this note is to elaborate on the following aspects of the structure of X^{an}:

(1) We define semistable vertex sets of X and their associated skeleta, which are finite metric graphs contained in X^{an}.

(2) We make explicit the bijective correspondence between the semistable vertex sets of X and the semistable models of X.

(3) We show that 'most' of X can be covered by skeleta as above, and use this fact to define a canonical metric on $\mathbb{H}(X^{\mathrm{an}}) := X^{\mathrm{an}} \setminus X(K)$ such that the resulting metric space is locally modeled on an $\mathbb{R}$-tree.

(4) We use results of Bosch and Lütkebohmert to prove Thuillier's non-Archimedean analogue of the Poincaré – Lelong formula. This includes the fact that the valuation of a nonzero rational function on X is a piecewise affine function on $\mathbb{H}(X^{\mathrm{an}})$, which is moreover *harmonic* in the sense that the sum of the outgoing slopes at any point is zero.

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The above results, while very useful and in large part well-known to experts, are often difficult or impossible to extract from the literature, although Ducros [**17**] is preparing a book on the subject. In [**2**] we apply these ideas to study the relationship between the analytification of a curve X and its tropicalization with respect to a rational map to a torus. In particular, we study the metric aspects of the tropicalization map and the relationship between skeleta and tropicalizations of X.

1.2. Skeleta and semistable models. The theory of skeleta goes back to Berkovich [**4, 6**], and is elaborated somewhat in the case of curves in Thuillier's (unpublished) thesis [**27**]. In Berkovich's approach, a skeleton is a subset of X^{an} which is associated to a semistable model of X. In contrast, we define the skeleton in terms of a *semistable vertex set* of X (called a *triangulation* in [**16**]; see also Temkin's lecture notes [**26**]), which is a finite set V of type-2 points of X^{an} such that $X^{\mathrm{an}} \setminus V$ is a disjoint union of open balls and finitely many open annuli. This approach has the advantage that it only makes reference to the analytic space X^{an} and is not conceptually tied to the semistable reduction theory of X, thus making certain constructions more natural.

If $V \subset X^{\mathrm{an}}$ is a semistable vertex set, the connected component decomposition

$$X^{\mathrm{an}} \setminus V = \coprod \mathbb{B}(1)_+ \amalg \coprod_{i=1}^{r} \mathbb{S}(a_i)_+$$

is called a *semistable decomposition* of X; here $\mathbb{B}(1)_+$ is the open unit ball and for $a_i \in \mathfrak{m}_R \setminus \{0\}$, we define $\mathbb{S}(a_i)_+$ to be the open annulus of inner radius $|a_i|$ and outer radius 1. A semistable decomposition of a non-Archimedean curve is somewhat analogous to a pair-of-pants decomposition of a Riemann surface. The annulus $\mathbb{S}(a_i)_+$ has a canonical closed subset $\Sigma(\mathbb{S}(a_i)_+)$, called its *skeleton*, which is identified with the open interval $\big(0, \mathrm{val}(a)\big)$. The *skeleton* of X associated to V is the set

$$\Sigma(X, V) := V \cup \bigcup_{i=1}^{r} \Sigma(\mathbb{S}(a_i)_+).$$

We show that $\Sigma(X, V)$ is naturally homeomorphic to a finite graph, with vertices V and open edges $\{\Sigma(\mathbb{S}(a_i)_+)\}_{i=1}^{r}$. Declaring the length of $\Sigma(\mathbb{S}(a_i)_+)$ to be $\mathrm{val}(a_i)$ (the logarithmic modulus of $\mathbb{S}(a_i)_+$) makes $\Sigma(X, V)$ into a metric graph. There is a deformation retraction $X^{\mathrm{an}} \to \Sigma(X, V)$, so $\Sigma(X, V)$ is connected. We give a relatively compelete account of these skeleta in §3.

1.3. Now let $\mathcal{X}$ be a semistable R-model of X. There is a *reduction map* $\mathrm{red}\colon X^{\mathrm{an}} \to \mathcal{X}_k$, defined as follows. Points of X^{an} correspond in a natural way to equivalence classes of maps $\mathrm{Spec}(L) \to X$, where L/K is a *valued* field extension. By the valuative criterion of properness, a point $x\colon \mathrm{Spec}(L) \to X$ extends in a unique way to a map $\mathrm{Spec}(\mathscr{O}_L) \to \mathcal{X}$ from the valuation ring of L; the reduction $\mathrm{red}(x)$ is defined to be the image of the closed point. The reduction map is anti-continuous, in that the inverse image of a Zariski-open set in $\mathcal{X}_k$ is a closed subset of X^{an} (and vice versa). The fibers of red are called *formal fibers*.

It follows from a theorem of Berkovich that for a generic point $\bar{x} \in \mathcal{X}_k$, the formal fiber $\mathrm{red}^{-1}(\bar{x})$ consists of a single type-2 point. If $\bar{x} \in \mathcal{X}_k$ is a closed point then by a theorem of Bosch and Lütkebohmert, the formal fiber $\mathrm{red}^{-1}(\bar{x})$ is isomorphic to an open ball or an open annulus if $\bar{x}$ is a smooth point or a node, respectively. It follows from this and the anti-continuity of red that the set $V_{\mathcal{X}}$ of

points of X^{an} reducing to generic points of $\mathcal{X}_k$ is a semistable vertex set, and that the decomposition

$$X^{\mathrm{an}} \setminus V = \coprod_{\bar{x} \in \mathcal{X}(k)} \mathrm{red}^{-1}(\bar{x})$$

of X^{an} into its formal fibers is a semistable decomposition. The associated skeleton $\Sigma(X, V_{\mathcal{X}})$ is the incidence graph of the irreducible components of $\mathcal{X}_k$.

We will give a proof of the following folklore theorem which says that the associaton $\mathcal{X} \mapsto V_{\mathcal{X}}$ is bijective, thus unifying the two notions of skeleta. See Theorem 4.11 and Proposition 4.10.

THEOREM. *The association $\mathcal{X} \mapsto V_{\mathcal{X}}$ is a bijective correspondence from the set of semistable models of X to the set of semistable vertex sets of X. Moreover, there exists a morphism of semistable models $\mathcal{X} \to \mathcal{X}'$ if and only if $V_{\mathcal{X}'} \subset V_{\mathcal{X}}$.*

1.4. In §5 we turn our attention to the metric nature of the analytic curve X^{an}. This is worked out 'by hand' in the case $X = \mathbb{P}^1$ by Baker and Rumely [**3**], but does not otherwise explicitly appear in the literature. We will prove that the metric graph structures on each skeleton of X^{an} are compatible, and that the resulting metric on their union extends by continuity to a unique metric on $\mathbb{H}(X^{\mathrm{an}}) = X^{\mathrm{an}} \setminus X(K)$. The resulting metric space is locally modeled on an $\mathbb{R}$-tree. At this point it is straightforward to give a proof of Thuillier's Poincaré–Lelong formula [**27**, Proposition 3.3.15] using classical machinery of Bosch and Lütkebohmert. We will do so without developing the harmonic analysis necessary to give the statement of Thuillier's theorem in terms of the non-Archimedean dd^c operator; for this reason we call the theorem the Slope Formula.

In the statement of the Slope Formula, we say that a function $F\colon \mathbb{H}(X^{\mathrm{an}}) \to \mathbb{R}$ is *piecewise affine with integer slopes* provided that, for every isometric embedding $\alpha\colon [a,b] \hookrightarrow \mathbb{H}(X^{\mathrm{an}})$, the composition $F \circ \alpha$ is a piecewise affine function with integer slopes on the interval $[a,b]$. We define the *set of tangent directions* T_x at a point $x \in \mathbb{H}(X^{\mathrm{an}})$ to be the set of germs of isometric embeddings $\alpha : [a,b] \hookrightarrow X^{\mathrm{an}}$ such that $\alpha(a) = x$, and we define the *outgoing slope of* F in the tangent direction v represented by α to be the right-hand derivative $d_v F(x)$ of $F \circ \alpha$ at a. One can extend these definitions to closed points $x \in X(K)$, although in our later formulation of the Slope Formula we avoid this issue.

THEOREM (Slope Formula). *Let $f \in K(X)^\times$ and let $F = -\log|f|\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$.*

(1) *F is piecewise affine with integer slopes on $\mathbb{H}(X^{\mathrm{an}})$.*
(2) *For $x \in \mathbb{H}(X^{\mathrm{an}})$ we have $d_v F(x) = 0$ for almost all $v \in T_x$, and*

$$\sum_{v \in T_x} d_v F(x) = 0.$$

In other words, F is harmonic.
(3) *For $x \in X(K)$ there is a unique tangent direction $v \in T_x$ and $d_v F(x) = \mathrm{ord}_x(f)$.*

See Theorem 5.15 for a more precise statement and proof.

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2. The skeleton of a generalized annulus

In this section we prove some preliminary facts about the building blocks of analytic curves, namely, open balls and open annuli. We also study punctured open balls in order to treat marked (or punctured) curves and their skeleta in §3.

2.1. Some analytic domains in $\mathbb{A}^1$. Define the extended tropicalization map, or valuation map,

$$\operatorname{trop}\colon \mathscr{M}(K[T]) = \mathbb{A}^{1,\mathrm{an}} \to \mathbb{R} \cup \{\infty\} \quad \text{by } \operatorname{trop}(\|\cdot\|) = -\log(\|T\|);$$

here $\mathscr{M}(\cdot)$ denotes the Berkovich spectrum. Clearly $\mathbb{G}_m^{\mathrm{an}} = \operatorname{trop}^{-1}(\mathbb{R})$. We use trop to define several analytic domains in $\mathbb{A}^{1,\mathrm{an}}$:

• For $a \in K^\times$ the *standard closed ball of radius* $|a|$ is $\mathbb{B}(a) = \operatorname{trop}^{-1}([\operatorname{val}(a), \infty])$. This is a polyhedral domain in the sense of [**23**]; more precisely, it is the affinoid domain with ring of analytic functions

$$K\langle a^{-1}t\rangle = \left\{ \sum_{n=0}^{\infty} a_n t^n : |a_n| \cdot |a|^n \to 0 \text{ as } n \to \infty \right\}.$$

The supremum norm is given by

$$\left| \sum_{n=0}^{\infty} a_n t^n \right|_{\sup} = \max\bigl\{ |a_n| \cdot |a|^n : n \geq 0 \bigr\}$$

and the canonical reduction is the polynomial ring $k[\tau]$, where τ is the residue of $a^{-1}t$.

• For $a \in K^\times$ the *standard open ball of radius* $|a|$ is $\mathbb{B}(a)_+ = \operatorname{trop}^{-1}\bigl((\operatorname{val}(a), \infty]\bigr)$. This is an open analytic domain which can be expressed as an increasing union of standard closed balls.

• For $a, b \in K^\times$ with $|a| \leq |b|$ the *standard closed annulus of inner radius* $|a|$ *and outer radius* $|b|$ is $\mathbb{S}(a,b) = \operatorname{trop}^{-1}([\operatorname{val}(b), \operatorname{val}(a)])$. This is a polytopal domain in $\mathbb{G}_m^{\mathrm{an}}$ [**20, 23**]; it is therefore an affinoid space whose ring of analytic functions is

$$\begin{aligned} &K\langle at^{-1}, b^{-1}t\rangle \\ &\quad = \left\{ \sum_{n=-\infty}^{\infty} a_n t^n : |a_n| \cdot |a|^n \to 0 \text{ as } n \to +\infty,\ |a_n| \cdot |b|^n \to 0 \text{ as } n \to -\infty \right\}. \end{aligned}$$

The supremum norm is given by

$$\left| \sum_{n=-\infty}^{\infty} a_n t^n \right|_{\sup} = \max\bigl\{ |a_n| \cdot |a|^n, |a_n| \cdot |b|^n : n \in \mathbb{Z} \bigr\}$$

and the canonical reduction is $k[\sigma, \tau]/(\sigma\tau - \overline{a/b})$, where σ (resp. τ) is the residue of at^{-1} (resp. $b^{-1}t$) and $\overline{a/b} \in k$ is the residue of a/b. The canonical reduction is an integral domain if and only if $|a| = |b|$, in which case the supremum norm is multiplicative. The *(logarithmic) modulus* of $\mathbb{S}(a,b)$ is by definition $\operatorname{val}(a) - \operatorname{val}(b)$.

• In the above situation, if $|a| \leq 1$ and $|b| = 1$ we write $\mathbb{S}(a) := \mathbb{S}(a, 1) = \operatorname{trop}^{-1}([0, \operatorname{val}(a)])$. In this case

$$K\langle at^{-1}, t\rangle \cong K\langle s, t\rangle/(st - a).$$

• For $a, b \in K^\times$ with $|a| < |b|$ the *standard open annulus of inner radius* $|a|$ *and outer radius* $|b|$ is $\mathbb{S}(a,b)_+ = \mathrm{trop}^{-1}\big((\mathrm{val}(b), \mathrm{val}(a))\big)$. This is an open analytic domain which can be expressed as an increasing union of standard closed annuli. The (*logarithmic*) *modulus* of $\mathbb{S}(a,b)_+$ is by definition $\mathrm{val}(a) - \mathrm{val}(b)$. As above we write $\mathbb{S}(a)_+ := \mathbb{S}(a,1)_+ = \mathrm{trop}^{-1}\big((0, \mathrm{val}(a))\big)$.

• For $a \in K^\times$ the *standard punctured open ball of radius* $|a|$ is $\mathbb{S}(0,a)_+ = \mathrm{trop}^{-1}\big((\mathrm{val}(a), \infty)\big)$, and the *standard punctured open ball of radius* $|a|^{-1}$ *around* ∞ is $\mathbb{S}(a,\infty)_+ = \mathrm{trop}^{-1}\big((-\infty, \mathrm{val}(a))\big)$. These are open analytic domains which can be written as an increasing union of standard closed annuli. By convention we define the *modulus* of $\mathbb{S}(0,a)_+$ and $\mathbb{S}(a,\infty)_+$ to be infinity. We write $\mathbb{S}(0)_+ = \mathbb{S}(0,1)_+$.

Note that if A is any of the above analytic domains in $\mathbb{A}^{1,\mathrm{an}}$ then $A = \mathrm{trop}^{-1}\big(\mathrm{trop}(A)\big)$. By a *standard generalized annulus* we will mean a standard closed annulus, a standard open annulus, or a standard punctured open ball, and by a *standard generalized open annulus* we will mean a standard open annulus or a standard punctured open ball. Note that by scaling we have isomorphisms

$$\mathbb{B}(a) \cong \mathbb{B}(1) \quad \mathbb{B}(a)_+ \cong \mathbb{B}(1)_+$$
$$\mathbb{S}(a,b) \cong \mathbb{S}(ab^{-1}) \quad \mathbb{S}(a,b)_+ \cong \mathbb{S}(ab^{-1})_+ \quad \mathbb{S}(0,a)_+ \cong \mathbb{S}(0)_+$$

and taking $t \mapsto t^{-1}$ yields $\mathbb{S}(1,\infty)_+ \cong \mathbb{S}(0,1)_+$.

Morphisms of standard closed annuli have the following structure:

Proposition 2.2. *Let $a \in R \setminus \{0\}$.*

(1) *The units in $K\langle at^{-1}, t\rangle$ are the functions of the form*

$$f(t) = \alpha\, t^d\big(1 + g(t)\big) \tag{2.1}$$

where $\alpha \in K^\times$, $d \in \mathbb{Z}$, and $|g|_{\sup} < 1$.

(2) *Let $f(t)$ be a unit as in* (2.1) *with $d > 0$ (resp. $d < 0$). The induced morphism $\phi\colon \mathbb{S}(a) \to \mathbb{G}_m^{\mathrm{an}}$ factors through a finite flat morphism $\mathbb{S}(a) \to \mathbb{S}(\alpha a^d, \alpha)$ (resp. $\mathbb{S}(a) \to \mathbb{S}(\alpha, \alpha a^d)$) of degree $|d|$.*

(3) *Let $f(t)$ be a unit as in* (2.1) *with $d = 0$. The induced morphism $\phi\colon \mathbb{S}(a) \to \mathbb{G}_m^{\mathrm{an}}$ factors through a morphism $\mathbb{S}(a) \to \mathbb{S}(\alpha, \alpha)$ which is not finite.*

Proof. The first assertion is proved in [**27**, Lemma 2.2.1] by considering the Newton polygon of $f(t)$. To prove (2) we easily reduce to the case $\alpha = 1$ and $d > 0$. Since $|f|_{\sup} = 1$ and $|f^{-1}|_{\sup} = |a|^{-d}$ the morphism ϕ factors set-theoretically through the affinoid domain $\mathbb{S}(a^d)$. Hence ϕ induces a morphism $\mathbb{S}(a) \to \mathbb{S}(a^d)$, so the homomorphism $K[s] \to K\langle at^{-1}, t\rangle$ extends to a homomorphism

$$F\colon K\langle a^d s^{-1}, s\rangle \to K\langle at^{-1}, t\rangle \qquad s \mapsto t^d\big(1 + g(t)\big), \quad a^d s^{-1} \mapsto (at^{-1})^d\big(1 + g(t)\big)^{-1}.$$

Since $|g|_{\sup} < 1$, the induced map on canonical reductions is

$$\widetilde{F}\colon k[\sigma_1, \sigma_2]/(\sigma_1\sigma_2 - \bar{a}^d) \to k[\tau_1, \tau_2]/(\tau_1\tau_2 - \bar{a}) \quad \sigma_i \mapsto \tau_i^d$$

where σ_1 (resp. σ_2, τ_1, τ_2) is the residue of $a^d s^{-1}$ (resp. s, at^{-1}, t). Now F is finite because $\widetilde{F}$ is finite [**8**, Theorem 6.3.5/1], and it is easy to see that F has degree d. Flatness of F is automatic because its source and target are principal ideal domains: any affinoid algebra is noetherian, and if $\mathscr{M}(\mathcal{A})$ is an affinoid subdomain of $\mathbb{A}^{1,\mathrm{an}} = \mathrm{Spec}(K[t])^{\mathrm{an}}$ then any maximal ideal of $\mathcal{A}$ is the extension of a maximal ideal of $K[t]$ by [**14**, Lemma 5.1.2(1)].

For (3), as above ϕ factors through $\mathbb{S}(1,1)$ if we assume $\alpha = 1$, so we get a homomorphism $F\colon K\langle a^d s^{-1}, s\rangle \to K\langle t, t^{-1}\rangle$. In this case the map $\widetilde{F}$ on canonical reductions is clearly not finite, so F is not finite. $\square$

2.3. The skeleton of a standard generalized annulus. Define a section $\sigma\colon \mathbb{R} \to \mathbb{G}_m^{\mathrm{an}}$ of the tropicalization map $\mathrm{trop}\colon \mathbb{G}_m^{\mathrm{an}} \to \mathbb{R}$ by

$$\sigma(r) = \|\cdot\|_r \quad \text{where} \quad \Big\| \sum_{n=-\infty}^{\infty} a_n t^n \Big\|_r = \max\big\{|a_n| \cdot \exp(-rn) : n \in \mathbb{Z}\big\}. \tag{2.2}$$

When $r \in G$ the point $\sigma(r)$ is the Shilov boundary point of the (strictly) affinoid domain $\mathrm{trop}^{-1}(r)$, and when $r \notin G$ we have $\mathrm{trop}^{-1}(r) = \{\sigma(r)\}$. The map σ is easily seen to be continuous, and is in fact the only continuous section of trop. We restrict σ to obtain continuous sections

$$\begin{aligned} [\mathrm{val}(b), \mathrm{val}(a)] &\to \mathbb{S}(a,b) & \big(\mathrm{val}(b), \mathrm{val}(a)\big) &\to \mathbb{S}(a,b)_+ \\ (\mathrm{val}(a), \infty) &\to \mathbb{S}(0,a)_+ & \big(-\infty, \mathrm{val}(a)\big) &\to \mathbb{S}(a,\infty)_+ \end{aligned}$$

of trop.

Definition. Let A be a standard generalized annulus. The *skeleton* of A is the closed subset

$$\Sigma(A) := \sigma(\mathbb{R}) \cap A = \sigma\big(\mathrm{trop}(A)\big).$$

More explicitly, the skeleton of $\mathbb{S}(a,b)$ (resp. $\mathbb{S}(a,b)_+$, resp. $\mathbb{S}(0,a)_+$, resp. $\mathbb{S}(a,\infty)_+$) is

$$\begin{aligned} \Sigma\big(\mathbb{S}(a,b)\big) &:= \sigma(\mathbb{R}) \cap \mathbb{S}(a,b) = \sigma([\mathrm{val}(b), \mathrm{val}(a)]) \\ \Sigma(\mathbb{S}(a,b)_+) &:= \sigma(\mathbb{R}) \cap \mathbb{S}(a,b)_+ = \sigma\big(\big(\mathrm{val}(b), \mathrm{val}(a)\big)\big) \\ \Sigma(\mathbb{S}(0,a)_+) &:= \sigma(\mathbb{R}) \cap \mathbb{S}(0,a)_+ = \sigma\big(\big(\mathrm{val}(a), \infty\big)\big) \\ \Sigma(\mathbb{S}(a,\infty)_+) &:= \sigma(\mathbb{R}) \cap \mathbb{S}(a,\infty)_+ = \sigma\big(\big(-\infty, \mathrm{val}(a)\big)\big). \end{aligned}$$

We identify $\Sigma(A)$ with the interval/ray $\mathrm{trop}(A)$ via trop or σ.

Note that $\tau_A := \sigma \circ \mathrm{trop}$ is a retraction of a standard generalized annulus A onto its skeleton. This can be shown to be a strong deformation retraction [**4**, Proposition 4.1.6]. Note also that the length of the skeleton of a standard generalized annulus is equal to its modulus.

The set-theoretic skeleton has the following intrinsic characterization:

Proposition 2.4 ([**27**, Proposition 2.2.5]). *The skeleton of a standard generalized annulus is the set of all points that do not admit an affinoid neighborhood isomorphic to* $\mathbb{B}(1)$.

The skeleton behaves well with respect to maps between standard generalized annuli:

Proposition 2.5. *Let A be a standard generalized annulus of nonzero modulus and let $\phi\colon A \to \mathbb{G}_m^{\mathrm{an}}$ be a morphism. Suppose that* $\mathrm{trop} \circ \phi\colon \Sigma(A) \to \mathbb{R}$ *is not constant. Then*:

(1) *For $x \in \Sigma(A)$ we have*

$$\mathrm{trop} \circ \phi(x) = d\,\mathrm{trop}(x) + \mathrm{val}(\alpha)$$

for some nonzero integer d and some $\alpha \in K^\times$.

(2) *Let* $B = \phi(A)$. *Then* $B = \operatorname{trop}^{-1}\big(\operatorname{trop}\big(\phi(A)\big)\big)$ *is a standard generalized annulus in* $\mathbb{G}_m^{\mathrm{an}}$ *of the same type, and* $\phi\colon A \to B$ *is a finite morphism of degree* $|d|$.

(3) $\phi\big(\Sigma(A)\big) = \Sigma(B)$ *and the following square commutes*:

$$\begin{tikzcd} \operatorname{trop}(A) \arrow[r, "d(\cdot)+\operatorname{val}(\alpha)"] \arrow[d, "\sigma"'] & \operatorname{trop}(B) \arrow[d, "\sigma"] \\ \Sigma(A) \arrow[r, "\phi"] & \Sigma(B) \end{tikzcd}$$

Proof. Let $A' \cong \mathbb{S}(a) \subset A$ be a standard closed annulus of nonzero modulus such that $\operatorname{trop}\circ\phi$ is not constant on $\Sigma(A')$. The morphism ϕ is determined by a unit $f \in K\langle at^{-1}, t\rangle^\times$, and for $x \in \Sigma(A')$ we have $\operatorname{trop}\big(\phi(x)\big) = -\log|f(x)|$. Writing $f(t) = \alpha\, t^d\big(1 + g(t)\big)$ as in (2.1), if $r = \operatorname{trop}(x)$ then $-\log|f(x)| = -\log\|f\|_r = dr + \operatorname{val}(\alpha)$ since $\|1+g\|_r = 1$. Since $\operatorname{trop}\circ\phi$ is nonconstant on $\Sigma(A')$ we must have $d \neq 0$. Part (1) follows by writing A as an increasing union of standard closed annuli and applying the same argument. The equality $B = \operatorname{trop}^{-1}\big(\operatorname{trop}\big(\phi(A)\big)\big)$ follows from Proposition 2.2(2) in the same way; since $\operatorname{trop}\big(\phi(A)\big)$ is a closed interval (resp. open interval, resp. open ray) when $\operatorname{trop}(A)$ is a closed interval (resp. open interval, resp. open ray), it follows that B is a standard generalized annulus of the same type as A.

For part (3) it suffices to show that $\phi\big(\sigma(r)\big) = \sigma\big(dr + \operatorname{val}(\alpha)\big)$ for $r \in \operatorname{trop}(A)$. This follows from the above because $\sigma\big(dr + \operatorname{val}(\alpha)\big)$ is the supremum norm on $\operatorname{trop}^{-1}\big(dr + \operatorname{val}(\alpha)\big)$ (when $r \in G$) and ϕ maps $\operatorname{trop}^{-1}(r)$ surjectively onto $\operatorname{trop}^{-1}\big(dr + \operatorname{val}(\alpha)\big)$. □

Corollary 2.6. *Let* $\phi\colon A_1 \to A_2$ *be a finite morphism of standard generalized annuli and let* d *be the degree of* ϕ. *Then* $\phi\big(\Sigma(A_1)\big) = \Sigma(A_2)$, $\phi\big(\sigma(r)\big) = \sigma\big(\pm dr + \operatorname{val}(\alpha)\big)$ *for all* $r \in \operatorname{trop}(A_1)$ *and some* $\alpha \in K^\times$, *and the modulus of* A_2 *is* d *times the modulus of* A_1. *In particular, two standard generalized annuli of the same type are isomorphic if and only if they have the same modulus.*

Proof. If the modulus of A_1 is zero then the result follows easily from Proposition 2.2. Suppose that the modulus of A_1 is nonzero. By Proposition 2.5, the only thing to show is that $\operatorname{trop}\circ\phi$ is not constant on $\Sigma(A)$. This is an immediate consequence of Proposition 2.2(3). □

2.7. General annuli and balls. In order to distinguish the properties of a standard generalized annulus and its skeleton that are invariant under isomorphism, it is convenient to make the following definition.

Definition. A *closed ball* (resp. *closed annulus*, resp. *open ball*, resp. *open annulus*, resp. *punctured open ball*) is a K-analytic space isomorphic to a standard closed ball (resp. standard closed annulus, resp. standard open ball, resp. standard open annulus, resp. standard punctured open ball). A *generalized annulus* is a closed annulus, an open annulus, or a punctured open ball, and a *generalized open annulus* is an open annulus or a punctured open ball.

2.8. Let A be a generalized annulus and fix an isomorphism $\phi\colon A \xrightarrow{\sim} A'$ with a standard generalized annulus A'. The *skeleton* of A is defined to be $\Sigma(A) := \phi^{-1}\big(\Sigma(A')\big)$. By Proposition 2.4 (or Corollary 2.6) this is a well-defined closed

subset of $\Sigma(A)$. We will view $\Sigma(A)$ as a closed interval (resp. open interval, resp. open ray) with endpoints in G, well-defined up to affine transformations of the form $r \mapsto \pm r + \mathrm{val}(\alpha)$ for $\alpha \in K^\times$. In particular $\Sigma(A)$ is naturally a metric space, and it makes sense to talk about piecewise affine-linear functions on $\Sigma(A)$ and of the slope of an affine-linear function on $\Sigma(A)$ up to sign. We remark that $G \cap \Sigma(A)$ is equal to the set of type-2 points of A contained in $\Sigma(A)$.

The retraction $\tau_{A'} = \sigma \circ \mathrm{trop}\colon A' \to \Sigma(A')$ induces a retraction $\tau_A\colon A \to \Sigma(A)$. By Proposition 2.5 this retraction is also independent of the choice of A'.

Definition 2.9. Let A be a generalized annulus, an open ball, or a closed ball. A *meromorphic function on* A is by definition a quotient of an analytic function on A by a nonzero analytic function on A.

Note that a meromorphic function f on A is analytic on the open analytic domain of A obtained by deleting the poles of f. If A is affinoid then f has only finitely many poles.

Let A be a generalized annulus, let $F\colon \Sigma(A) \to \mathbb{R}$ be a piecewise affine function, and let x be contained in the interior of $\Sigma(A)$. The *change of slope of* F *at* x is defined to be

$$\lim_{\epsilon \to 0} \big(F'(x+\epsilon) - F'(x-\epsilon)\big);$$

this is independent of the choice of identification of $\Sigma(A)$ with an interval in $\mathbb{R}$.

We will need the following special case of the Slope Formula (Theorem 5.15). Its proof is an easy Newton polygon computation.

Proposition 2.10. *Let A be a generalized annulus, let f be a meromorphic function on A, and define $F\colon \Sigma(A) \to \mathbb{R}$ by $F(x) = -\log|f(x)|$.*

(1) *F is a piecewise affine function with integer slopes, and for x in the interior of $\Sigma(A)$ the change of slope of F at x is equal to the number of poles of f retracting to x minus the number of zeros of f retracting to x, counted with multiplicity.*

(2) *Suppose that $A = \mathbb{S}(0)_+$ and that f extends to a meromorphic function on $\mathbb{B}(1)_+$. Then for all $r \in (0,\infty)$ such that $r > \mathrm{val}(y)$ for all zeros and poles y of f in A, we have $F'(r) = \mathrm{ord}_0(f)$.*

Corollary 2.11. *Let f be an analytic function on $\mathbb{S}(0)_+$ that extends to a meromorphic function on $\mathbb{B}(1)_+$ with a pole at 0 of order d. Suppose that f has fewer than d zeros on $\mathbb{S}(0)_+$. Then $F = \log|f|$ is a monotonically increasing function on $\Sigma(\mathbb{S}(0)_+) = (0,\infty)$.*

The following facts will also be useful:

Lemma 2.12. *Let A be a generalized annulus. Then the open analytic domain $A \setminus \Sigma(A)$ is isomorphic to an infinite disjoint union of open balls. Each connected component B of $A \setminus \Sigma(A)$ retracts onto a single point $x \in \Sigma(A)$, and the closure of B in A is equal to $B \cup \{x\}$.*

Proof. First we assume that A is the standard closed annulus $\mathbb{S}(1) = \mathscr{M}(K\langle t^{\pm 1}\rangle)$ of modulus zero. Then $\Sigma(A) = \{x\}$ is the Shilov boundary point of A. The canonical reduction of A is isomorphic to $\mathbb{G}_{m,k}$, the inverse image of the generic point of $\mathbb{G}_{m,k}$ is x, the inverse image of a residue class $\bar{y} \in k^\times = \mathbb{G}_m(k)$ is the open ball $\{\|\cdot\| : \|t-y\| < 1\}$ (where $y \in R^\times$ reduces to $\bar{y}$), and the fibers over the closed points of $\mathbb{G}_{m,k}$ are the connected components of $A \setminus \{x\}$ by [**27**, Lemme 2.1.13].

This proves the first assertion, and the second follows from the anti-continuity of the reduction map.

Now let A be any generalized annulus; we may assume that A is standard. Let $r \in \mathrm{trop}(A)$. If $r \notin G$ then $\mathrm{trop}^{-1}(r)$ is a single point of type 3, so suppose $r \in G$, say $r = \mathrm{val}(a)$ for $a \in K^\times$. After translating by a^{-1} we may and do assume that $r = 0$, so $\mathrm{trop}^{-1}(r) = \mathbb{S}(1) = \mathscr{M}(K\langle t^{\pm 1}\rangle)$. The subset $\mathbb{S}(1) \setminus \{\sigma(0)\}$ is clearly closed in $A \setminus \Sigma(A)$, and it is open as well since it is the union of the open balls $\{\|\cdot\| : \|t-y\| < 1\}$ for $y \in R^\times$. Therefore the connected components of $\mathbb{S}(1)\setminus\{\sigma(0)\}$ are also connected components of $A \setminus \Sigma(A)$, so we are reduced to the case treated above. □

Lemma 2.13. *Let A be a generalized annulus and let f be a unit on A. Then $x \mapsto \log|f(x)|$ factors through the retraction $\tau_A \colon A \to \Sigma(A)$. In particular, $x \mapsto \log|f(x)|$ is locally constant away from $\Sigma(A)$.*

Proof. This follows immediately from Lemma 2.12 and the elementary fact that a unit on an open ball has constant absolute value. □

3. Semistable decompositions and skeleta of curves

For the rest of this paper X denotes a smooth connected algebraic curve over K, $\widehat{X}$ denotes its smooth completion, and $D = \widehat{X} \setminus X$ denotes the set of punctures. We will define a skeleton inside of X relative to the following data.

Definition 3.1. A *semistable vertex set of* $\widehat{X}$ is a finite set V of type-2 points of $\widehat{X}^{\mathrm{an}}$ such that $\widehat{X}^{\mathrm{an}} \setminus V$ is a disjoint union of open balls and finitely many open annuli. A *semistable vertex set of* X is a semistable vertex set of $\widehat{X}$ such that the punctures in D are contained in distinct connected components of $\widehat{X}^{\mathrm{an}} \setminus V$ isomorphic to open balls. A decomposition of X^{an} into a semistable vertex set and a disjoint union of open balls and finitely many generalized open annuli is called a *semistable decomposition of* X.

When we refer to 'an open ball in a semistable decomposition of X' or 'a generalized open annulus in a semistable decomposition of X' we will always mean a connected component of $X^{\mathrm{an}} \setminus V$ of the specified type. Note that the punctured open balls in a semistable decomposition of X are in bijection with D, and that there are no punctured open balls in a semistable decomposition of a complete curve. A semistable vertex set of X is also a semistable vertex set of $\widehat{X}$.

The semistable vertex sets of $\widehat{X}$ correspond naturally and bijectively to isomorphism classes of semistable formal models of $\hat{X}$. See Theorem 4.11.

Lemma 3.2. *Let V be a semistable vertex set of X, let A be a connected component of $X^{\mathrm{an}} \setminus V$, and let $\bar{A}$ be the closure of A in $\widehat{X}^{\mathrm{an}}$. Let $\partial_{\lim} A = \bar{A} \setminus A$ be the* limit boundary *of A, i.e., the set of limit points of A in $\widehat{X}^{\mathrm{an}}$ that are not contained in A.*[1]

(1) *If A is an open ball then $\partial_{\lim} A = \{x\}$ for some $x \in V$.*

(2) *Suppose that A is an open annulus, and fix an isomorphism $A \cong \mathbb{S}(a)_+$. Let $r = \mathrm{val}(a)$. Then $\sigma \colon (0,r) \to A$ extends in a unique way to a continuous map $\sigma : [0,r] \to X^{\mathrm{an}}$ such that $\sigma(0), \sigma(r) \in V$, and $\partial_{\lim} A = \{\sigma(0), \sigma(r)\}$. (It may happen that $\sigma(0) = \sigma(r)$.)*

[1] As opposed to the canonical boundary discussed in [**4**, §2.5.7].

(3) *Suppose that A is a punctured open ball, and fix an isomorphism $A \cong \mathbb{S}(0)_+$. Then $\sigma\colon (0,\infty) \to A$ extends in a unique way to a continuous map $\sigma\colon [0,\infty] \to \widehat{X}^{\mathrm{an}}$ such that $\sigma(0) \in V$, $\sigma(\infty) \in D$, and $\partial_{\lim} A = \{\sigma(0), \sigma(\infty)\}$.*

Proof. First note that in (1) and (2), $\bar{A}$ is the closure of A in X^{an} because every point of $\widehat{X} \setminus X$ has an open neighborhood disjoint from A. Since A is closed in $X \setminus V$, its limit boundary is contained in V.

Suppose that A is an open ball, and fix an isomorphism $\phi\colon \mathbb{B}(1)_+ \xrightarrow{\sim} A$. For $r \in (0,\infty)$ we define $\|\cdot\|_r \in \mathbb{B}(1)_+$ by (2.2). Fix an affine open subset X' of X such that $A \subset (X')^{\mathrm{an}}$. For any $f \in K[X']$ the map $r \mapsto \log\|f\|_r$ is piecewise affine with finitely many changes in slope by Proposition 2.10. Therefore we may define $\|f\| = \lim_{r\to 0}\|f\|_r \in \mathbb{R}$. The map $f \mapsto \|f\|$ is easily seen to be a multiplicative norm on $K[X']$, hence defines a point $x \in (X')^{\mathrm{an}} \subset X^{\mathrm{an}}$.

Let y be the Shilov point of $\mathbb{B}(1)$ and let $A' = \mathbb{B}(1)_+ \cup \{y\}$. Since $\mathbb{B}(1) \setminus \{y\}$ is a disjoint union of open balls it is clear that A' is a closed, hence compact subset of $\mathbb{B}(1)$. Extend ϕ to a map $A' \to (X')^{\mathrm{an}} \subset X^{\mathrm{an}}$ by $\phi(y) = x$. We claim that ϕ is continuous. By the definition of the topology on $(X')^{\mathrm{an}}$ it suffices to show that the set $U = \{z \in A' : |f(\phi(z))| \in (c_1, c_2)\}$ is open for all $f \in K[X']$ and all $c_1 < c_2$. Since $U \cap \mathbb{B}(1)_+$ is open, we need to show that U contains a neighborhood of y if $y \in U$, i.e., if $\|f\| \in (c_1, c_2)$. Choose $a \in \mathfrak{m}_R \setminus \{0\}$ such that f has no zeros in $\mathbb{S}(a)_+$. Note that $\mathbb{S}(a)_+ \cup \{y\}$ is a neighborhood of y in A'. Since $\|f\| = \lim_{r\to 0}\|f\|_r$ we have that $\|f\|_r \in (c_1, c_2)$ for r close enough to 1; hence we may shrink $\mathbb{S}(a)_+$ so that $\phi(\Sigma(\mathbb{S}(a)_+)) \subset (r_1, r_2)$. With Lemma 2.13 this implies that $\mathbb{S}(a)_+ \subset U$, so ϕ is indeed continuous. Since A' is compact we have that $\phi(A') = A \cup \{x\}$ is closed, which completes the proof of (1).

If $A \cong \mathbb{S}(0)_+$ is a punctured open ball then certainly the puncture 0 is in $\bar{A}$. The above argument effectively proves the rest of (3), and (2) is proved in exactly the same way. □

Definition 3.3. Let V be a semistable vertex set of X. The *skeleton* of X with respect to V is

$$\Sigma(X, V) = V \cup \bigcup \Sigma(A)$$

where A runs over all of the connected components of $X^{\mathrm{an}} \setminus V$ that are generalized open annuli.

Lemma 3.4. *Let V be a semistable vertex set of X and let $\Sigma = \Sigma(X, V)$ be the associated skeleton. Then*:

(1) *Σ is a closed subset of X^{an} which is compact if and only if $X = \widehat{X}$.*

(2) *The limit boundary of Σ in $\widehat{X}^{\mathrm{an}}$ is equal to D.*

(3) *The connected components of $X^{\mathrm{an}} \setminus \Sigma(X, V)$ are open balls, and the limit boundary $\partial_{\lim} B$ of any connected component B is a single point $x \in \Sigma(X, V)$.*

(4) *Σ is equal to the set of points in X^{an} that do not admit an affinoid neighborhood isomorphic to $\mathbb{B}(1)$ and disjoint from V.*

Proof. The first two assertions are clear from Lemma 3.2, and the third follows from Lemmas 3.2 and 2.12. Let Σ' be the set of points in X^{an} that do not admit an affinoid neighborhood isomorphic to $\mathbb{B}(1)$ and disjoint from V. We have $\Sigma' \subset \Sigma$ by (3). For the other inclusion, let $x \in \Sigma$. If $x \in V$ then clearly $x \in \Sigma'$, so suppose $x \notin V$. Then the connected component A of x in $X^{\mathrm{an}} \setminus V$ is a generalized

open annulus; since any connected neighborhood of x is contained in A, we have $x \in \Sigma'$ by Proposition 2.4. □

DEFINITION 3.5. Let V be a semistable vertex set of X. The *completed skeleton* of X with respect to V is defined to be the closure of $\Sigma(X,V)$ in $\widehat{X}^{\mathrm{an}}$ and is denoted $\widehat{\Sigma}(X,V)$, so $\widehat{\Sigma}(X,V) = \Sigma(X,V) \cup D$. The completed skeleton has the structure of a graph with vertices $V \cup D$; the interiors of the edges of $\Sigma(X,V)$ are the skeleta of the generalized open annuli in the semistable decomposition of X coming from V.

REMARK 3.6. By Lemma 3.4(1), if $X = \widehat{X}$ then the skeleton $\Sigma(X,V) = \widehat{\Sigma}(X,V)$ is a finite metric graph (cf. §4.9). If X is not proper then $\widehat{\Sigma}(X,V)$ is a finite graph with vertex set $V \cup D$, but the "metric" on $\widehat{\Sigma}(X,V)$ is degenerate since it has edges of infinite length. See Definition 3.10.

DEFINITION 3.7. Let V be a semistable vertex set of X and let $\Sigma = \Sigma(X,V)$. We define a retraction $\tau_V = \tau_\Sigma \colon X^{\mathrm{an}} \to \Sigma$ as follows. Let $x \in X^{\mathrm{an}} \setminus \Sigma$ and let B_x be the connected component of x in $X^{\mathrm{an}} \setminus \Sigma$. Then $\partial_{\mathrm{lim}}(B_x) = \{y\}$ for a single point $y \in X^{\mathrm{an}}$; we set $\tau_V(x) = y$.

LEMMA 3.8. *Let V be a semistable vertex set of X. The retraction $\tau_V \colon X^{\mathrm{an}} \to \Sigma(X,V)$ is continuous, and if A is a generalized open annulus in the semistable decomposition of X then τ_V restricts to the retraction $\tau_A \colon A \to \Sigma(A)$ defined in* §2.8.

PROOF. The second assertion follows from Lemma 2.12, so τ_V is continuous when restricted to any connected component A of $X^{\mathrm{an}} \setminus V$ which is a generalized open annulus. Hence it is enough to show that if $x \in V$ and U is an open neighborhood of x then $\tau_V^{-1}(U)$ contains an open neighborhood of x. This is left as an exercise to the reader. □

PROPOSITION 3.9. *Let V be a semistable vertex set of X. Then $\Sigma(X,V)$ and $\widehat{\Sigma}(X,V)$ are connected.*

PROOF. This follows from the continuity of τ_V and the connectedness of X^{an}. □

The skeleton of a curve naturally carries the following kind of combinatorial structure, which is similar to that of a metric graph.

DEFINITION 3.10. A *dimension*-1 *abstract G-affine polyhedral complex* is a combinatorial object Σ consisting of the following data. We are given a finite discrete set V of *vertices* and a collection of finitely many *segments* and *rays*, where a segment is a closed interval in $\mathbb{R}$ with distinct endpoints in G and a ray is a closed ray in $\mathbb{R}$ with endpoint in G. Segments and rays are only defined up to isometries of $\mathbb{R}$ of the form $r \mapsto \pm r + \alpha$ for $\alpha \in G$. The segments and rays are collectively called *edges* of Σ. Finally, we are given an identification of the endpoints of the edges of Σ with vertices. The complex Σ has an obvious realization as a topological space, which we will also denote by Σ. If Σ is connected then it is a metric space under the shortest-path metric.

A *morphism* of dimension-1 abstract G-affine polyhedral complexes is a continuous function $\phi \colon \Sigma \to \Sigma'$ sending vertices to vertices and such that if $e \subset \Sigma$ is an edge then either $\phi(e)$ is a vertex of Σ', or $\phi(e)$ is an edge of Σ' and for all $r \in e$ we have $\phi(r) = dr + \alpha$ for a nonzero integer d and some $\alpha \in G$.

A *refinement* of a dimension-1 abstract G-affine polyhedral complex is a complex Σ' obtained from Σ by inserting vertices at G-points of edges of Σ and dividing those edges in the obvious way. Note that Σ and Σ' have the same topological and metric space realizations.

REMARK 3.11. Abstract integral G-affine polyhedral complexes of arbitrary dimension are defined in [**27**, §1] in terms of groups of integer-slope G-affine functions. In the one-dimensional case the objects of loc. cit. are roughly the same as the dimension-1 abstract integral G-affine polyhedral complexes in the sense of our ad-hoc definition above, since the knowledge of what functions on a line segment have slope one is basically the same as the data of a metric. We choose to use this definition for concreteness and in order to emphasize the metric nature of these objects.

3.12. Let V be a semistable vertex set of X. Then $\Sigma(X,V)$ is a dimension-1 abstract G-affine polyhedral complex with vertex set V whose edges are the closures of the skeleta of the generalized open annuli in the semistable decomposition of X. In particular, $\Sigma(X,V)$ is a metric space, and each edge e of $\Sigma(X,V)$ is identified via a local isometry with the skeleton of the corresponding generalized open annulus. Note that if e is a segment then the length of e is equal to the modulus of the corresponding open annulus. The G-points of $\Sigma(X,V)$ are exactly the type-2 points of X contained in $\Sigma(X,V)$.

PROPOSITION 3.13. *Let V be a semistable vertex set of X and let X' be a nonempty open subscheme of X.*

(1) *Let V' be a semistable vertex set of X' containing V. Then $\Sigma(X,V) \subset \Sigma(X',V')$ and $\Sigma(X',V')$ induces a refinement of $\Sigma(X,V)$. Furthermore, $\tau_{\Sigma(X,V)} \circ \tau_{\Sigma(X',V')} = \tau_{\Sigma(X,V)}$.*

(2) *Let $V' \subset \Sigma(X,V)$ be a finite set of type-2 points. Then $V \cup V'$ is a semistable vertex set of X and $\Sigma(X, V \cup V')$ is a refinement of $\Sigma(X,V)$.*

(3) *Let $W \subset X^{\mathrm{an}}$ be a finite set of type-2 points. Then there is a semistable vertex set V' of X' containing $V \cup W$.*

PROOF. In (1), the inclusion $\Sigma(X,V) \subset \Sigma(X',V')$ follows from Lemma 3.4(4), and the fact that $\Sigma(X',V')$ induces a refinement of $\Sigma(X,V)$ is an easy consequence of the structure of morphisms of generalized open annuli (Proposition 2.5). The equality $\tau_{\Sigma(X,V)} \circ \tau_{\Sigma(X',V')} = \tau_{\Sigma(X,V)}$ follows from the definitions. In (2) we may assume that V' is a single point; it then suffices to show that if A is a generalized open annulus and $x \in \Sigma(A)$ is a type-2 point then $A \setminus \{x\}$ is a disjoint union of generalized open annuli and open balls. Choose an identification of A with a standard generalized open annulus $\mathbb{S}(a,b)_+$ such that $0 \in \big(\mathrm{trop}(b), \mathrm{trop}(a)\big)$ and $x = \sigma(0)$. As in the proof of Lemma 2.12 we have that $\mathbb{S}(1) \setminus \{x\}$ is a disjoint union of open balls, and is open and closed in A; hence

$$A \setminus \{x\} = \mathbb{S}(a,1)_+ \amalg (\mathbb{S}(1) \setminus \{x\}) \amalg \mathbb{S}(1,b)_+$$

is a disjoint union of generalized open annuli and open balls.

It suffices to prove (3) when $W = \{x\}$ and $X = X'$, and when $W = \varnothing$ and $X \setminus X' = \{y\}$. In the first case, we may assume that $x \notin \Sigma(X,V)$ by the above. Suppose that the connected component A of $X^{\mathrm{an}} \setminus V$ containing x is an open ball. One shows as in (2) that $A \setminus \{x\}$ is a disjoint union of open balls and an open annulus, so $V \cup \{x\}$ is a semistable vertex set. If A is a generalized open annulus

then the connected component of x in $A \setminus \{\tau_A(x)\}$ is an open ball, so $V \cup \{x, \tau_A(x)\}$ is a semistable vertex set. In the case $X \setminus X' = \{y\}$ one proceeds in exactly the same way. □

A semistable vertex set V is called *strongly semistable* if the graph $\widehat{\Sigma}(X, V)$ has no loop edges. (See Definition 3.5.)

COROLLARY 3.14. *Any semistable vertex set of X is contained in a strongly semistable vertex set of X.*

4. Relation with semistable models

Recall that X is a smooth connected algebraic curve over K, $\widehat{X}$ is its smooth completion, and $D = \widehat{X} \setminus X$ is the set of punctures. The (formal) semistable reduction theory of a smooth complete algebraic curve was worked out carefully in [**10**] in the language of rigid analytic spaces and formal analytic varieties (see Remark 4.2(3)); one can view much of this section as a translation of that paper into our language of semistable vertex sets.

4.1. It is more natural in the context of analytic geometry to use Bosch-Lütkebohmert's theory [**9**] of admissible formal models of $\widehat{X}^{\mathrm{an}}$ instead of algebraic models of $\widehat{X}$. An *admissible R-algebra* is an R-flat quotient of a convergent power series ring

$$R\langle X_1, \ldots, X_n\rangle = \left\{\sum a_I X^I \in R[\![X_1, \ldots, X_n]\!] : |a_I| \to 0 \text{ as } |I| \to \infty\right\}$$

by a finitely generated ideal. An *admissible formal R-scheme* is a formal scheme $\mathfrak{X}$, locally modeled on the formal spectrum of an admissible R-algebra. If A is an admissible R-algebra then $A \otimes_R K$ is a strictly K-affinoid algebra, and the functor taking A to its Berkovich spectrum $\mathscr{M}(A \otimes_R K)$ glues to give the *Raynaud generic fiber functor* $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{an}}$ from the category of admissible formal R-schemes to the category of K-analytic spaces. There is a canonical anti-continuous reduction map $\mathrm{red}\colon \mathfrak{X}^{\mathrm{an}} \to \mathfrak{X}_k$.

DEFINITION. (1) A connected reduced algebraic curve over a field k is called *semistable* if its singularities are ordinary double points. It is *strongly semistable* if in addition its irreducible components are smooth.

(2) A *(strongly) semistable formal R-curve* is an integral admissible formal R-curve $\mathfrak{X}$ whose special fiber is a (strongly) semistable curve. A *(strongly) semistable formal model* for $\widehat{X}$ is a (strongly) semistable proper formal R-curve $\mathfrak{X}$ equipped with an isomorphism $\mathfrak{X}^{\mathrm{an}} \cong \widehat{X}^{\mathrm{an}}$.

REMARK 4.2. (1) Let $\mathfrak{X}$ be a (strongly) semistable formal R-curve. Then $\mathfrak{X}$ is proper if and only if $\mathfrak{X}^{\mathrm{an}}$ is proper by [**24**, Corollary 4.4]. Therefore the properness hypothesis in the definition of a (strongly) semistable formal model for $\widehat{X}$ is redundant.

(2) A (*strongly*) *semistable algebraic model* for $\widehat{X}$ is a flat and integral proper relative curve $\mathcal{X} \to \mathrm{Spec}(R)$ whose special fiber is a (strongly) semistable curve and whose generic fiber is equipped with an isomorphism to $\widehat{X}$. A (strongly) semistable algebraic model $\mathcal{X}$ for $\widehat{X}$ gives rise to a (strongly) semistable formal model $\mathfrak{X}$ for $\widehat{X}$ by completing. Indeed, $\mathfrak{X}$ and $\mathcal{X}$ have the same special fiber, and $\mathfrak{X}^{\mathrm{an}} \cong \widehat{X}^{\mathrm{an}}$ by [**14**, Theorem 5.3.1(4)]. Conversely, a (strongly) semistable formal model for $\widehat{X}$

uniquely algebraizes to a (strongly) semistable algebraic model by a suitable formal GAGA theorem over R [**1**, Corollary 2.3.19]. Hence there is no essential difference between the algebraic and formal semistable reduction theories of $\widehat{X}$.

(3) Let $\mathfrak{X}$ be a semistable formal R-curve. Since $\overline{\mathfrak{X}}$ is reduced, $\mathfrak{X}$ is a formal analytic variety in the sense of [**10**]. In particular, if $\mathrm{Spf}(A)$ is a formal affine open subset of $\mathfrak{X}$ then A is the ring of power-bounded elements of $A \otimes_R K$ — that is, $\mathrm{Spf}(A)$ is the canonical model of $\mathscr{M}(A \otimes_R K)$ — and $A \otimes_R k$ is the canonical reduction of $A \otimes_R K$.

Let $a \in R \setminus \{0\}$. The *standard formal annulus of modulus* $\mathrm{val}(a)$ is defined to be

$$\mathfrak{S}(a) \coloneqq \mathrm{Spf}\left(R\langle s,t\rangle/(st-a)\right).$$

This is the canonical model of the standard closed annulus $\mathbb{S}(a)$.

Proposition 4.3. *Let $\mathfrak{X}$ be a strongly semistable formal R-curve and let $\xi \in \mathfrak{X}$ be a singular point of $\overline{\mathfrak{X}}$. There is a formal neighborhood $\mathfrak{U}$ of ξ and an étale morphism $\phi\colon \mathfrak{U} \to \mathfrak{S}(a)$ for some $a \in \mathfrak{m}_R \setminus \{0\}$ such that ϕ^{an} restricts to an isomorphism* $\mathrm{red}^{-1}(\xi) \xrightarrow{\sim} \mathbb{S}(a)_+$.

Proof. This is essentially [**10**, Proposition 2.3]; here we explain how the proof of loc. cit. implies the proposition. Shrinking $\mathfrak{X}$ if necessary, we may and do assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine and connected, and that the maximal ideal $\mathfrak{m}_\xi \subset \bar{A} = A \otimes_R k$ corresponding to ξ is generated by two functions $\bar{f}, \bar{g} \in \bar{A}$ whose product is zero (this is possible because $\mathfrak{X}$ is *strongly* semistable). Let $k[x,y]/(xy) \to \bar{A}$ be the homomorphism sending $x \mapsto \bar{f}$ and $y \mapsto \bar{g}$. Since ξ is an ordinary double point, we can choose $\bar{f}$ and $\bar{g}$ such that the map on completed local rings $k[\![x,y]\!]/(xy) \to \widehat{\mathscr{O}}_{\overline{\mathfrak{X}},\xi}$ is an isomorphism. It follows from [**18**, Proposition 17.6.3] that the morphism $\mathrm{Spec}(\bar{A}) \to \mathrm{Spec}\big(k[x,y]/(xy)\big)$ is étale at ξ, so shrinking $\mathfrak{X}$ further we may assume that $\mathrm{Spec}(\bar{A}) \to \mathrm{Spec}\big(k[x,y]/(xy)\big)$ is étale. One then proceeds as in the proof of [**10**, Proposition 2.3] to find lifts $f, g \in A$ of $\bar{f}, \bar{g}$ such that $fg = a \in R \setminus \{0\}$; the induced morphism $\mathfrak{X} \to \mathfrak{S}(a)$ is étale because it lifts an étale morphism on the special fiber. The fact that ϕ restricts to an isomorphism $\mathrm{red}^{-1}(\xi) \xrightarrow{\sim} \mathfrak{S}(a)_+$ is part (i) of loc. cit. □

The following characterization of strongly semistable formal R-curves is also commonly used in the literature, for example in [**27**, Definition 2.2.8] (see also Remark 2.2.9 in loc. cit.).

Corollary 4.4. *An integral admissible formal R-curve $\mathfrak{X}$ is strongly semistable if and only if it has a covering by Zariski-open sets $\mathfrak{U}$ which admit an étale morphism to $\mathfrak{S}(a_{\mathfrak{U}})$ for some $a_{\mathfrak{U}} \in R \setminus \{0\}$.*

Let $\mathfrak{X}$ and $\mathfrak{X}'$ be two semistable formal models for $\widehat{X}$. We say that $\mathfrak{X}$ *dominates* $\mathfrak{X}'$, and we write $\mathfrak{X} \geq \mathfrak{X}'$, if there exists an R-morphism $\mathfrak{X} \to \mathfrak{X}'$ inducing the identity on the generic fiber $\widehat{X}^{\mathrm{an}}$. Such a morphism is unique if it exists. The relation $\geq$ is a partial ordering on the set of semistable formal models for $\widehat{X}$. (We will always consider semistable formal models of $\widehat{X}$ up to isomorphism; any isomorphism is unique.)

4.5. Semistable models and semistable decompositions. The special fiber of the canonical model for $\mathbb{B}(1)$ is isomorphic to $\mathbb{A}^1_k$, and the inverse image of the origin is the open unit ball $\mathbb{B}(1)_+$. When $|a| < 1$ the special fiber of $\mathfrak{S}(a)$ is isomorphic to $k[x,y]/(xy)$, and the inverse image of the origin under the reduction map is $\mathbb{S}(a)_+$. The following much stronger version of these facts provides the relation between semistable models and semistable decompositions of $\widehat{X}$.

THEOREM 4.6 (Berkovich, Bosch – Lütkebohmert). *Let $\mathfrak{X}$ be an integral admissible formal R-curve with reduced special fiber and let $\xi \in \overline{\mathfrak{X}}$ be any point.*

(1) *ξ is a generic point if and only if $\mathrm{red}^{-1}(\xi)$ is a single type-2 point of $\mathfrak{X}^{\mathrm{an}}$.*
(2) *ξ is a smooth closed point if and only if $\mathrm{red}^{-1}(\xi) \cong \mathbb{B}(1)_+$.*
(3) *ξ is an ordinary double point if and only if $\mathrm{red}^{-1}(\xi) \cong \mathbb{S}(a)_+$ for some $a \in \mathfrak{m}_R \setminus \{0\}$.*

PROOF. As in Remark 4.2(3) the hypothesis on the special fiber of $\mathfrak{X}$ allows us to view $\mathfrak{X}$ as a formal analytic variety. Hence the first statement follows from [**4**, Proposition 2.4.4], and the remaining assertions are [**10**, Propositions 2.2 and 2.3]. □

Let $\mathfrak{X}$ be a semistable formal model for $\widehat{X}$. We let $V(\mathfrak{X})$ denote the inverse image of the set of generic points of $\overline{\mathfrak{X}}$ under the reduction map. This is a finite set of type-2 points of $\mathfrak{X}^{\mathrm{an}}$ that maps bijectively onto the set of generic points of $\overline{\mathfrak{X}}$.

COROLLARY 4.7. *Let $\mathfrak{X}$ be a semistable formal model for $\widehat{X}$. Then $V(\mathfrak{X})$ is a semistable vertex set of $\widehat{X}$, and the decomposition of $\widehat{X}^{\mathrm{an}} \setminus V(\mathfrak{X})$ into formal fibers is a semistable decomposition.*

PROOF. By [**27**, Lemma 2.1.13] the formal fibers of $\mathfrak{X}$ are the connected components of $\widehat{X}^{\mathrm{an}} \setminus V(\mathfrak{X})$, so the assertion reduces to Theorem 4.6. □

REMARK 4.8. The semistable vertex set $V(\mathfrak{X})$ is a semistable vertex set of the punctured curve $X = \widehat{X} \setminus D$ if and only if the punctures $x \in D$ reduce to distinct smooth closed points of $\mathfrak{X}(k)$, that is, if and only if $\mathfrak{X}$ is a semistable model of the marked curve $(\widehat{X}, D)$.

4.9. Let $\mathfrak{X}$ be a semistable formal model for $\widehat{X}$. Let $\xi \in \overline{\mathfrak{X}}$ be a singular point and let $z_1, z_2 \in \widehat{X}^{\mathrm{an}}$ be the inverse images of the generic points of $\overline{\mathfrak{X}}$ specializing to ξ (it may be that $z_1 = z_2$). Then z_1, z_2 are the vertices of the edge in $\Sigma(\widehat{X}, V(\mathfrak{X}))$ whose interior is $\Sigma(\mathrm{red}^{-1}(\xi))$ by the anti-continuity of the reduction map and Lemma 3.2(2). It follows that $\Sigma(\widehat{X}, V(\mathfrak{X}))$ is the incidence graph of $\overline{\mathfrak{X}}$ (cf. Remark 3.6). In other words, the vertices of $\Sigma\big(\widehat{X}, V(\mathfrak{X})\big)$ correspond to irreducible components of $\overline{\mathfrak{X}}$ and the edges of $\Sigma\big(\widehat{X}, V(\mathfrak{X})\big)$ correspond to the points where the components of $\overline{\mathfrak{X}}$ intersect. Moreover, if $\mathfrak{X}$ admits an étale map to some $\mathfrak{S}(a) = \mathrm{Spf}\big(R\langle x,y\rangle/(xy-a)\big)$ in a neighborhood of ξ, then $\mathrm{val}(a)$ is the length of the edge corresponding to ξ (see the proof of Proposition 4.10).

It is clear from the above that a semistable formal model $\mathfrak{X}$ for $\widehat{X}$ is strongly semistable if and only if $V(\mathfrak{X})$ is a strongly semistable vertex set.

Berkovich [**7**] and Thuillier [**27**] define the skeleton of a *strongly* semistable formal R-curve using Proposition 4.3. In order to use their results, we must show that the two notions of the skeleton agree:

PROPOSITION 4.10. *Let $\mathfrak{X}$ be a strongly semistable formal model for $\widehat{X}$. The skeleton $\Sigma\big(\widehat{X}, V(\mathfrak{X})\big)$ is naturally identified with the skeleton of $\mathfrak{X}$ defined in* [**27**] *as dimension-1 abstract G-affine polyhedral complexes.*

PROOF. Thuillier [**27**, Definition 2.2.13] defines the skeleton $S(\mathfrak{X})$ of $\mathfrak{X}$ to be the set of all points that do not admit an affinoid neighborhood isomorphic to $\mathbb{B}(1)$ and disjoint from $V(\mathfrak{X})$, so $\Sigma\big(\widehat{X}, V(\mathfrak{X})\big) = S(\mathfrak{X})$ as sets by Lemma 3.4(4). Let $\xi \in \overline{\mathfrak{X}}$ be a singular point and let $\mathfrak{U}$ be a formal affine neighborhood of ξ admitting an étale morphism $\phi\colon \mathfrak{U} \to \mathfrak{S}(a)$ and inducing an isomorphism $\mathrm{red}^{-1}(\xi) \xrightarrow{\sim} \mathbb{S}(a)_+$ as in Proposition 4.3. Shrinking $\mathfrak{U}$ if necessary, we may and do assume that ξ is the only singular point of $\overline{\mathfrak{U}}$ and that $\overline{\mathfrak{U}}$ has two generic points ζ_1, ζ_2. Let $z_1, z_2 \in V(\mathfrak{X})$ be the inverse images of ζ_1, ζ_2. Then $\Sigma\big(X, V(\mathfrak{X})\big) \cap \mathfrak{U}^{\mathrm{an}}$ is the edge in $\Sigma\big(X, V(\mathfrak{X})\big)$ connecting z_1, z_2 with interior $\Sigma(\mathrm{red}^{-1}(\xi))$. Since ϕ^{an} maps $\mathrm{red}^{-1}(\xi)$ isomorphically onto $\mathbb{S}(a)_+$ it induces an isometry $\Sigma\big(X, V(\mathfrak{X})\big) \cap \mathfrak{U}^{\mathrm{an}} \xrightarrow{\sim} \Sigma\big(\mathbb{S}(a)\big)$. The polyhedral structure on $S(\mathfrak{X}) \cap \mathfrak{U}^{\mathrm{an}}$ is more or less by definition induced by the identification of $\Sigma\big(\mathbb{S}(a)\big)$ with $[0, \mathrm{val}(a)]$; see [**27**, Theorem 2.2.10]. Hence $\Sigma\big(\widehat{X}, V(\mathfrak{X})\big) = S(\mathfrak{X})$ as G-affine polyhedral complexes. □

In order to prove that semistable vertex sets are in one-to-one correspondence with semistable models as above, it remains to construct a semistable model from a semistable decomposition. The following theorem is folklore; while it is well-known to experts, and in some sense is implicit in [**25**], we have been unable to find an explicit reference.

THEOREM 4.11. *The association $\mathfrak{X} \mapsto V(\mathfrak{X})$ sets up a bijection between the set of semistable formal models of $\widehat{X}$ (up to isomorphism) and the set of semistable vertex sets of $\widehat{X}$. Furthermore, $\mathfrak{X}$ dominates $\mathfrak{X}'$ if and only if $V(\mathfrak{X}') \subset V(\mathfrak{X})$.*

We will need the following lemmas in the proof of Theorem 4.11.

LEMMA 4.12. (1) *Let $B \subset \widehat{X}^{\mathrm{an}}$ be an analytic open subset isomorphic to an open ball. Then $\widehat{X}^{\mathrm{an}} \setminus B$ is an affinoid domain in $\widehat{X}^{\mathrm{an}}$.*
(2) *Let $A \subset \widehat{X}^{\mathrm{an}}$ be an analytic open subset isomorphic to an open annulus. Then $\widehat{X}^{\mathrm{an}} \setminus A$ is an affinoid domain in $\widehat{X}^{\mathrm{an}}$.*

PROOF. First we establish (1). Let us fix an isomorphism $B \cong \mathbb{B}(1)_+$. By [**10**, Lemma 3.5(c)], for any $a \in K^\times$ with $|a| < 1$ the compact set $\widehat{X}^{\mathrm{an}} \setminus \mathbb{B}(a)_+$ is an affinoid domain in $\widehat{X}^{\mathrm{an}}$. The limit boundary of $\widehat{X}^{\mathrm{an}} \setminus \mathbb{B}(a)_+$ in $\widehat{X}^{\mathrm{an}}$ is the Gauss point $\|\cdot\|_{\mathrm{val}(a)}$ of $\mathbb{B}(a)$; this coincides with the Shilov boundary of $\widehat{X}^{\mathrm{an}} \setminus \mathbb{B}(a)_+$ by [**27**, Proposition 2.1.12]. The proof of Lemma 3.2 shows that $\partial_{\lim}(B) = \{x\}$ where $x = \lim_{r\to 0}\|\cdot\|_r$.

By the Riemann–Roch theorem, there exists a meromorphic function on $\widehat{X}$ which is regular away from $0 \in \mathbb{B}(1)_+$ and which has a zero outside of $\mathbb{B}(1)_+$. Fix such a function f, and scale it so that $|f(x)| = 1$. By Corollary 2.11, the function $F(y) = -\log|f(y)|$ is a monotonically decreasing function on $\Sigma(\mathbb{S}(0)_+) \cong (0, \infty)$ such that $\lim_{r\to 0} F(\|\cdot\|_r) = 0$. The meromorphic function f defines a finite morphism $\phi\colon \widehat{X} \to \mathbb{P}^1$, which analytifies to a finite morphism $\phi^{\mathrm{an}}\colon \widehat{X}^{\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$. Let $Y = \{y \in \widehat{X}^{\mathrm{an}} : |f(y)| \le 1\}$ be the inverse image of $\mathbb{B}(1) \subset \mathbb{P}^{1,\mathrm{an}}$ under ϕ^{an}, so Y is an affinoid domain in $\widehat{X}^{\mathrm{an}}$. For $a \in \mathfrak{m}_R \setminus \{0\}$ the point $\|\cdot\|_{\mathrm{val}(a)}$ is the Shilov boundary of $\widehat{X}^{\mathrm{an}} \setminus \mathbb{B}(a)_+$, so $|f| \le \|f\|_{\mathrm{val}(a)}$ on $\widehat{X}^{\mathrm{an}} \setminus \mathbb{B}(a)_+$. Since

$\widehat{X}^{\mathrm{an}} \setminus B \subset \widehat{X}^{\mathrm{an}} \setminus \mathbb{B}(a)_+$ for all $a \in \mathfrak{m}_R \setminus \{0\}$ we have $|f| \leq \lim_{r\to 0}\|f\|_r = 1$ on $\widehat{X}^{\mathrm{an}} \setminus B$. Therefore $\widehat{X}^{\mathrm{an}} \setminus B \subset Y$.

We claim that $\widehat{X}^{\mathrm{an}} \setminus B$ is a connected component of Y. Clearly it is closed in Y. Since f has finitely many zeros in B, there exists $a \in \mathfrak{m}_R \setminus \{0\}$ such that f is a unit on $\mathbb{S}(a)_+ \subset \mathbb{B}(1)_+$. By Lemma 2.13 we have that $|f| > 1$ on $\mathbb{S}(a)_+$, so $\widehat{X}^{\mathrm{an}} \setminus B = (\widehat{X}^{\mathrm{an}} \setminus \mathbb{B}(a)) \cap Y$ is open in Y. Hence $\widehat{X}^{\mathrm{an}} \setminus \mathbb{B}(1)_+$ is affinoid, being a connected component of the affinoid domain Y.

We will reduce the second assertion to the first by doing surgery on $\widehat{X}^{\mathrm{an}}$, following the proof of [**5**, Proposition 3.6.1]. Let A_1 be a closed annulus inside of A, so $A \setminus A_1 \cong \mathbb{S}(a)_+ \amalg \mathbb{S}(b)_+$ for $a, b \in \mathfrak{m}_R \setminus \{0\}$. Let $(X')^{\mathrm{an}}$ be the analytic curve obtained by gluing $\widehat{X}^{\mathrm{an}} \setminus A_1$ to two copies of $\mathbb{B}(1)_+$ along the inclusions $\mathbb{S}(a)_+ \hookrightarrow \mathbb{B}(1)_+$ and $\mathbb{S}(b)_+ \hookrightarrow \mathbb{B}(1)_+$. One verifies easily that $(X')^{\mathrm{an}}$ is proper in the sense of [**4**, §3], so $(X')^{\mathrm{an}}$ is the analytification of a unique algebraic curve X'. By construction $\widehat{X}^{\mathrm{an}} \setminus A$ is identified with the affinoid domain $(X')^{\mathrm{an}} \setminus (\mathbb{B}(1)_+ \amalg \mathbb{B}(1)_+)$ in $(X')^{\mathrm{an}}$, so we can apply (1) twice to $(X')^{\mathrm{an}}$ to obtain the result. □

Remark 4.13. Let $\mathscr{U}$ be an affinoid domain in $\widehat{X}^{\mathrm{an}}$ and let x be a Shilov boundary point of $\mathscr{U}$. Since $\widetilde{\mathscr{H}(x)}$, the residue field of the completed residue field $\mathscr{H}(x)$ at x, is isomorphic to the function field of an irreducible component of the canonical reduction of $\mathscr{U}$, the point x has type 2. Hence Lemma 4.12 implies that if $A \subset \widehat{X}^{\mathrm{an}}$ is an open ball or an open annulus then $\partial_{\lim}(A)$ consists of either one or two type-2 points of $\widehat{X}^{\mathrm{an}}$ since $\partial_{\lim}(A)$ is the Shilov boundary of $\widehat{X}^{\mathrm{an}} \setminus A$.

Recall that if V is a semistable vertex set of $\widehat{X}$ then there is a retraction $\tau_V = \tau_{\Sigma(\widehat{X},V)} \colon \widehat{X}^{\mathrm{an}} \to \Sigma(\widehat{X}, V)$.

Lemma 4.14. *Let V be a semistable vertex set of $\widehat{X}$ and let $x \in V$. Then there are infinitely many open balls in the semistable decomposition for $\widehat{X}$ which retract to x.*

Proof. Suppose that there is at least one edge of $\Sigma(\widehat{X}, V)$. Deleting all of the open annuli in the semistable decomposition of $\widehat{X}$ yields an affinoid domain Y by Lemma 4.12. The set $\tau_V^{-1}(x)$ is a connected component of Y, so $\tau_V^{-1}(x)$ is an affinoid domain as well. The Shilov boundary of $\tau_V^{-1}(x)$ agrees with its limit boundary $\{x\}$ in $\widehat{X}^{\mathrm{an}}$; by construction $\tau_V^{-1}(x) \setminus \{x\}$ is a disjoint union of open balls, which are the formal fibers of the canonical model of $\tau_V^{-1}(x)$ by [**27**, Lemma 2.1.13]. Any nonempty curve over k has infinitely many points, so $\tau_V^{-1}(x) \setminus \{x\}$ is a disjoint union of infinitely many open balls.

If $\Sigma(\widehat{X}, V)$ has no edges then $\widehat{X}^{\mathrm{an}} \setminus \{x\}$ is a disjoint union of open balls. Deleting one of these balls yields an affinoid domain by Lemma 4.12, and the above argument goes through. □

4.15. Proof of Theorem 4.11. First we prove that $\mathfrak{X} \mapsto V(\mathfrak{X})$ is surjective, i.e., that any semistable vertex set comes from a semistable formal model. Let V be a semistable vertex set of $\widehat{X}$, let $\Sigma = \Sigma(\widehat{X}, V)$, and let $\tau = \tau_\Sigma \colon \widehat{X}^{\mathrm{an}} \to \Sigma$ be the retraction.

4.15.1. *Case* 1. Suppose that Σ has at least two edges. Let e be an edge in Σ, let $A_0, A_1, \ldots, A_r$ $(r \geq 1)$ be the open annuli in the semistable decomposition of $\widehat{X}$, and suppose that $\Sigma(A_0)$ is the interior of e. Then $\widehat{X} \setminus (\bigcup_{i=1}^r A_i)$ is an affinoid

domain by Lemma 4.12, and $\tau^{-1}(e)$ is a connected component of $\widehat{X} \setminus (\bigcup_{i=1}^r A_i)$. Hence $\tau^{-1}(e)$ is an affinoid domain in $\widehat{X}^{\mathrm{an}}$. Let $\mathfrak{Y}$ be its canonical model. Let $x, y \in \widehat{X}^{\mathrm{an}}$ be the endpoints of e, so $\{x, y\} = \partial_{\lim}\big(\tau^{-1}(e)\big)$ is the Shilov boundary of $\tau^{-1}(e)$, and $\tau^{-1}(e) \setminus \{x, y\}$ is a disjoint union of open balls and the open annulus A_0. By [**27**, Lemma 2.1.13], the formal fibers of $\tau^{-1}(e) \to \overline{\mathfrak{Y}}$ are the connected components of $\tau^{-1}(e) \setminus \{x, y\}$, so $\overline{\mathfrak{Y}}$ has either one or two irreducible components (depending on whether $x = y$) which intersect along a single ordinary double point ξ by Theorem 4.6. Let $\mathfrak{C}_x$ (resp. $\mathfrak{C}_y$) be the irreducible component of $\mathfrak{Y}$ whose generic point is the reduction of x (resp. y). Using the anti-continuity of the reduction map one sees that $\mathrm{red}^{-1}(\overline{\mathfrak{C}}_x \setminus \{\xi\}) = \tau^{-1}(x)$ and $\mathrm{red}^{-1}(\overline{\mathfrak{C}}_y \setminus \{\xi\}) = \tau^{-1}(y)$. It follows that the formal affine subset $\mathfrak{C}_x \setminus \{\xi\}$ (resp. $\mathfrak{C}_y \setminus \{\xi\}$) is the canonical model of the affinoid domain $\tau^{-1}(x)$ (resp. $\tau^{-1}(y)$).

Applying the above for every edge e of Σ allows us to glue the canonical models of the affinoid domains $\tau^{-1}(e)$ together along the canonical models of the affinoid domains $\tau^{-1}(x)$ corresponding to the vertices x of Σ. Thus we obtain a semistable formal model $\mathfrak{X}$ of $\widehat{X}$ such that $V(\mathfrak{X}) = V$ (cf. Remark 4.2(1)).

4.15.2. *Case* 2. Suppose that Σ has one edge e and two vertices x, y. Let B_x, B'_x (resp. B_y, B'_y) be distinct open balls in the semistable decomposition of $\widehat{X}$ retracting to x (resp. y), so $Y \coloneqq \widehat{X}^{\mathrm{an}} \setminus (B_x \cup B_y)$ and $Y' \coloneqq \widehat{X}^{\mathrm{an}} \setminus (B'_x \cup B'_y)$ are affinoid domains by Lemma 4.12. Let $\mathfrak{Y}$ (resp. $\mathfrak{Y}'$) be the canonical model of Y (resp. Y'). Arguing as in Case 1 above, $\overline{\mathfrak{Y}}$ and $\overline{\mathfrak{Y}}'$ are affine curves with two irreducible components intersecting along a single ordinary double point ξ. Furthermore, $Z = Y \cap Y'$ is an affinoid domain whose canonical model $\mathfrak{Z}$ is obtained from $\mathfrak{Y}$ (resp. $\mathfrak{Y}'$) by deleting one smooth point from each component. Gluing $\mathfrak{Y}$ to $\mathfrak{Y}'$ along $\mathfrak{Z}$ yields the desired semistable formal model $\mathfrak{X}$ of $\widehat{X}$.

4.15.3. *Case* 3. Suppose that Σ has just one vertex x. Let B, B' be distinct open balls in the semistable decomposition of $\widehat{X}$, let $Y = \widehat{X}^{\mathrm{an}} \setminus B$, let $Y' = \widehat{X}^{\mathrm{an}} \setminus B'$, and let $Z = Y \cap Y'$. Gluing the canonical models of Y and Y' along the canonical model of Z gives us our semistable formal model as in Case 2.

4.15.4. A semistable formal model of $\widehat{X}$ is determined by its formal fibers [**10**, Lemma 3.10], so $\mathfrak{X} \mapsto V(\mathfrak{X})$ is bijective. It remains to prove that $\mathfrak{X}$ dominates $\mathfrak{X}'$ if and only if $V(\mathfrak{X}') \subset V(\mathfrak{X})$. If $\mathfrak{X}$ dominates $\mathfrak{X}'$ then $V(\mathfrak{X}') \subset V(\mathfrak{X})$ by the surjectivity and functoriality of the reduction map. Conversely let V, V' be semistable vertex sets of $\widehat{X}$ such that $V' \subset V$. The corresponding semistable formal models $\mathfrak{X}, \mathfrak{X}'$ were constructed above by finding coverings $\mathscr{U}$, $\mathscr{U}'$ of $\widehat{X}^{\mathrm{an}}$ by affinoid domains whose canonical models glue along the canonical models of their intersections. (Such a covering is called a *formal covering* in [**10**].) It is clear that if $\mathscr{U}$ refines $\mathscr{U}'$, in the sense that every affinoid in $\mathscr{U}$ is contained in an affinoid in $\mathscr{U}'$, then we obtain a morphism $\mathfrak{X} \to \mathfrak{X}'$ of semistable formal models. Therefore it suffices to show that we can choose $\mathscr{U}, \mathscr{U}'$ such that $\mathscr{U}$ refines $\mathscr{U}'$ when $V' \subset V$ in all of the cases treated above. We will carry out this procedure in the situation of Case 1, when V is the union of V' with a type-2 point $x \in \Sigma' = \Sigma(\widehat{X}, V')$ not contained in V'; the other cases are similar and are left to the reader (cf. the proof of Proposition 3.13).

In the situation of Case 1, the formal covering corresponding to V' is the set

$$\mathscr{U}' = \{\tau^{-1}(e) : e \text{ is an edge of } \Sigma'\}.$$

By Proposition 3.13(2) the skeleton $\Sigma = \Sigma(\widehat{X}, V)$ is a refinement of Σ', obtained by subdividing the edge e_0 containing x to allow x as a vertex. Let e_1, e_2 be the edges of Σ containing x. Then $\tau^{-1}(e_1), \tau^{-1}(e_2)$ are affinoid domains in $\widehat{X}^{\mathrm{an}}$ contained in $\tau^{-1}(e_0)$, so the formal covering $\mathscr{U} = \{\tau^{-1}(e) : e \text{ is an edge of } \Sigma\}$ is a refinement of $\mathscr{U}'$, as desired. □

4.16. Stable models and the minimal skeleton. Here we explain when and in what sense there exists a minimal semistable vertex set of X. Of course this question essentially reduces to the existence of a stable model of X when $X = \widehat{X}$; using [**10**] we can also treat the case when X is not proper.

Definition. Let $x \in X^{\mathrm{an}}$ be a type-2 point. The *genus* of x, denoted $g(x)$, is defined to be the genus of the smooth proper connected k-curve with function field $\widetilde{\mathscr{H}}(x)$, the residue field of the completed residue field $\mathscr{H}(x)$ at x.

Remark 4.17. Let V be a semistable vertex set of $\widehat{X}$ and let $x \in \widehat{X}^{\mathrm{an}}$ be a type-2 point with positive genus. Then $x \in V$, since otherwise x admits a neighborhood which is isomorphic to an analytic domain in $\mathbb{P}^{1,\mathrm{an}}$ and the genus of any type-2 point in $\mathbb{P}^{1,\mathrm{an}}$ is zero.

Remark 4.18. Let $\mathfrak{X}$ be a semistable formal model for $\widehat{X}$, let $x \in V(\mathfrak{X})$, and let $\overline{\mathfrak{C}} \subset \overline{\mathfrak{X}}$ be the irreducible component with generic point $\zeta = \mathrm{red}(x)$. Then $\widetilde{\mathscr{H}}(x)$ is isomorphic to $\mathscr{O}_{\overline{\mathfrak{X}},\zeta}$ by [**4**, Proposition 2.4.4], so $g(x)$ is the genus of the normalization of $\overline{\mathfrak{C}}$. It follows from [**10**, Theorem 4.6] that

$$g(\widehat{X}) = \sum_{x \in V(\mathfrak{X})} g(x) + g\big(\Sigma(\widehat{X}, V)\big) \tag{4.1}$$

where $g(\widehat{X})$ is the genus of $\widehat{X}$ and $g\big(\Sigma(\widehat{X}, V)\big) = \mathrm{rank}_{\mathbb{Z}}\big(H_1(\Sigma(\widehat{X}, V), \mathbb{Z})\big)$ is the genus of $\Sigma(\widehat{X}, V)$ as a topological space (otherwise known as the cyclomatic number of the graph $\Sigma(\widehat{X}, V)$). The important equation (4.1) is known as the *genus formula*.

Definition 4.19. The *Euler characteristic of* X is defined to be

$$\chi(X) = 2 - 2g(\widehat{X}) - \#D.$$

Definition 4.20. A semistable vertex set V of X is *stable* if there is no $x \in V$ of genus zero and valence less than three in $\Sigma(X, V)$. We call the corresponding semistable decomposition of X *stable* as well. A semistable formal model $\mathfrak{X}$ of $\widehat{X}$ such that $V(\mathfrak{X})$ is a stable vertex set of $\widehat{X}$ is called a *stable formal model*.

A semistable vertex set V of X is *minimal* if V does not properly contain a semistable vertex set V'. Any semistable vertex set contains a minimal one.

Proposition 4.21. *Let V be a semistable vertex set of X and let $x \in V$ be a point of genus zero.*

(1) *Suppose that x has valence one in $\widehat{\Sigma}(X, V)$, let e be the edge adjoining x, and let y be the other endpoint of e. If $y \notin D$ then $V \setminus \{x\}$ is a semistable vertex set of X and $\widehat{\Sigma}(X, V \setminus \{x\})$ is the graph obtained from $\widehat{\Sigma}(X, V)$ by removing x and the interior of e.*

(2) *Suppose that x has valence two in $\widehat{\Sigma}(X, V)$, let e_1, e_2 be the edges adjoining x, and let x_1 (resp. x_2) be the other endpoint of e_1 (resp. e_2). If $\{x_1, x_2\} \not\subset D$ then*

$V \setminus \{x\}$ is a semistable vertex set of X and $\widehat{\Sigma}(X, V \setminus \{x\})$ is the graph obtained from $\widehat{\Sigma}(X, V)$ by joining e_1, e_2 into a single edge.

PROOF. This is essentially [**10**, Lemma 6.1] translated into our language. □

By a *topological vertex* of a finite connected graph Γ we mean a vertex of valence at least 3. The set of topological vertices only depends on the topological realization of Γ.

THEOREM 4.22 (Stable reduction theorem). *There exists a semistable vertex set of X. If V is a minimal semistable vertex set of X then:*

(1) *If $\chi(X) \leq 0$ then $\Sigma(X, V)$ is the set of points in X^{an} that do not admit an affinoid neighborhood isomorphic to $\mathbb{B}(1)$.*
(2) *If $\chi(X) < 0$ then V is stable and*

$$V = \{x \in \Sigma(X, V) : x \text{ is a topological vertex of } \Sigma(X, V) \text{ or } g(x) > 0\}.$$

COROLLARY 4.23. *If $\chi(X) \leq 0$ then there is a unique set-theoretic minimal skeleton of X, and if $\chi(X) < 0$ then there is a unique stable vertex set of X.*

PROOF OF THEOREM 4.22. The existence of a semistable vertex set of $\widehat{X}$ follows from the classical theorem of Deligne and Mumford [**15**] as proved analytically (over a non-noetherian rank-1 valuation ring) in [**10**, Theorem 7.1]. The existence of a semistable vertex set of X then follows from Proposition 3.13(3). Let V be a minimal semistable vertex set of X and let $\Sigma = \Sigma(X, V)$. If $\chi(X) < 0$ then one applies Proposition 4.21 in the standard way to prove the second assertion, and if $\chi(X) \leq 0$ then Proposition 4.21(1) guarantees that every genus-zero vertex of Σ has valence at least two.

Suppose that $\chi(X) \leq 0$. Let Σ' be the set of points of X^{an} that do not admit an affinoid neighborhood isomorphic to $\mathbb{B}(1)$. By Lemma 3.4(4) we have $\Sigma' \subset \Sigma$. Let $x \in \Sigma$, and suppose that x admits an affinoid neighborhood U isomorphic to $\mathbb{B}(1)$. We will show by way of contradiction that Σ has a vertex of valence less than two in U (any vertex contained in U has genus zero); in fact we will show that $\Sigma \cap U$ is a tree. Let y be the Gauss point of U. If $y \in \Sigma$ then we may replace V by $V \cup \{y\}$ by Proposition 3.13(2) to assume that $y \in V$. Since U is closed and any connected component of $X^{\mathrm{an}} \setminus V$ that intersects U is contained in U, the retraction $\tau_\Sigma \colon X^{\mathrm{an}} \to \Sigma$ restricts to a retraction $U \to U \cap \Sigma$. Since U is contractible, $U \cap \Sigma$ is a tree as claimed. □

REMARK 4.24. If $\chi(X) = 0$ then either $g(X) = 0$ and $\#D = 2$ or $g(X) = 1$ and $\#D = 0$. In the first case, the skeleton of $X \cong \mathbb{G}_m$ is the line connecting 0 and ∞, and any type-2 point on this line is a minimal semistable vertex set. In the second case, $X = \widehat{X}$ is an elliptic curve with respect to some choice of distinguished point $0 \in X(K)$. If X has good reduction then there is a unique point $x \in X^{\mathrm{an}}$ with $g(x) = 1$; in this case $\{x\}$ is the unique stable vertex set of X and $\Sigma(X, \{x\}) = \{x\}$.

Suppose now that $(X, 0)$ is an elliptic curve with multiplicative reduction, i.e., X is a Tate curve. By Tate's uniformization theory [**8**, §9.7], there is a unique $q = q_X \in K^\times$ with $\mathrm{val}(q) > 0$ and an étale morphism $u \colon \mathbb{G}_m^{\mathrm{an}} \to X^{\mathrm{an}}$ which is a homomorphism of group objects (in the category of K-analytic spaces) with kernel $u^{-1}(0) = q^{\mathbb{Z}}$. For brevity we will often write $X^{\mathrm{an}} \cong \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}$. The so-called Tate parameter q is related to the j-invariant $j = j_X$ of X in such a way that $\mathrm{val}(q) = -\mathrm{val}(j)$ (it is the q-expansion of the modular function j). Let Z be the

retraction of the set $q^{\mathbb{Z}}$ onto the skeleton of $\mathbb{G}_m$, i.e., the collection of Gauss points of the balls $\mathbb{B}(q^n)$ for $n \in \mathbb{Z}$. Then $\mathbb{G}_m^{\mathrm{an}} \setminus Z$ is the disjoint union of the open annuli $\{\mathbb{S}(q^{n+1}, q^n)_+\}_{n\in\mathbb{Z}}$ and infinitely many open balls, and every connected component of $\mathbb{G}_m^{\mathrm{an}} \setminus Z$ maps isomorphically onto its image in X^{an}. It follows that $X^{\mathrm{an}} \setminus \{u(1)\}$ is a disjoint union of an open annulus A isomorphic to $\mathbb{S}(q)_+$ and infinitely many open balls. Hence $V = \{u(1)\}$ is a (minimal) semistable vertex set of X, and the associated (minimal) skeleton Σ is a circle of circumference $\mathrm{val}(q) = -\mathrm{val}(j_E)$. We have $u(1) = \tau_\Sigma(0)$, so any type-2 point on Σ is a minimal semistable vertex set, as any such point is the retraction of a K-point of X (which we could have chosen to be 0).

See also [**19**, Example 7.20].

Remark 4.25. Given a smooth complete curve $\widehat{X}/K$ of genus g and a subset D of 'marked points' of $\widehat{X}(K)$ satisfying the inequality $2 - 2g - n \leq 0$, where $n = \#D \geq 0$, one obtains a canonical pair (Γ, w) consisting of an abstract metric graph and a vertex weight function, where $\Gamma = \Sigma(\hat{X} \setminus D, V)$ is the minimal skeleton of $\widehat{X} \setminus D$ and $w : \Gamma \to \mathbb{Z}_{\geq 0}$ takes $x \in \Gamma$ to 0 if $x \notin V$ and to $g(x)$ if $x \in V$. (A closely related construction can be found in [**28**, §2].) If $2 - 2g - n < 0$, this gives a canonical 'abstract tropicalization map' $\mathrm{trop}\colon M_{g,n} \to M_{g,n}^{\mathrm{trop}}$, where $M_{g,n}^{\mathrm{trop}}$ is the moduli space of n-pointed tropical curves of genus g as defined, for example, in [**13**, §3]. The map $\mathrm{trop}\colon M_{g,n} \to M_{g,n}^{\mathrm{trop}}$ is certainly deserving of further study.

4.26. Application to the local structure theory of X. The semistable reduction theorem and its translation into the language of semistable vertex sets yields the following information about the local structure theory of an analytic curve. (Conversely, one can study the local structure of an analytic curve directly and derive the semistable reduction theorem: see [**25**].)

Corollary 4.27. *Let $x \in X^{\mathrm{an}}$. There is a fundamental system of open neighborhoods $\{U_\alpha\}$ of x of the following form*:

(1) *If x is a type-1 or a type-4 point then the U_α are open balls.*

(2) *If x is a type-3 point then the U_α are open annuli with $x \in \Sigma(U_\alpha)$.*

(3) *If x is a type-2 point then $U_\alpha = \tau_V^{-1}(W_\alpha)$ where W_α is a simply-connected open neighborhood of x in $\Sigma(X, V)$ for some semistable vertex set V of X containing x, and each $U_\alpha \setminus \{x\}$ is a disjoint union of open balls and open annuli.*

Proof. Since X has a semistable decomposition, if x is a point of type 1, 3, or 4 then x has a neighborhood isomorphic to an open annulus or an open ball. Hence we may assume that $X = \mathbb{P}^1$ and $x \in \mathbb{B}(1)_+$. By [**3**, Proposition 1.6] the set of open balls with finitely many closed balls removed forms a basis for the topology on $\mathbb{B}(1)_+$; assertions (1) and (2) follow easily from this.

Let f be a meromorphic function on X; deleting the zeros and poles of f, we may assume that f is a unit on X. Let $F = \log|f|\colon X^{\mathrm{an}} \to \mathbb{R}$ and let $U = F^{-1}\big((a, b)\big)$ for some interval $(a, b) \subset \mathbb{R}$. Let x be a type-2 point contained in U. Since such U form a sub-basis for the topology on X^{an} it suffices to prove that there is a neighborhood of x of the form described in (3) contained in U. Let V be a semistable vertex set for X containing x. By Proposition 2.5 and Lemma 2.13 we have that F is affine-linear on the edges of $\Sigma(X, V)$ and that F factors through $\tau_V\colon X^{\mathrm{an}} \to \Sigma(X, V)$. Therefore if W is any simply-connected neighborhood of x in $\Sigma(X, V)$ contained in $U = F^{-1}\big((a, b)\big)$ then $\tau_V^{-1}(W) \subset U$. If we assume in addition

that the intersection of W with any edge of Σ adjoining x is a half-open interval with endpoints in G then $\tau_V^{-1}(W) \setminus \{x\}$ is a disjoint union of open balls and open annuli. $\square$

DEFINITION 4.28. A neighborhood of $x \in X^{\mathrm{an}}$ of the form described in Corollary 4.27 is called a *simple neighborhood* of x.

4.29. A simple neighborhood of a type-2 point $x \in X^{\mathrm{an}}$ has the following alternative description. Let V be a semistable vertex set containing x and let W be a simply-connected neighborhood of x in $\Sigma(X, V)$ such that the intersection of W with any edge adjoining x is a half-open interval with endpoints in G, so $U = \tau_V^{-1}(W)$ is a simple neighborhood of x. Adding the boundary of W to V, we may assume that the connected components of $U \setminus \{x\}$ are connected components of $X^{\mathrm{an}} \setminus V$. Let $\mathfrak{X}$ be the semistable formal model of $\widehat{X}$ associated to V and let $\overline{\mathfrak{C}} \subset \overline{\mathfrak{X}}$ be the irreducible component with generic point $\mathrm{red}(x)$. Since W contains no loop edges of $\Sigma(X, V)$, the component $\overline{\mathfrak{C}}$ is smooth. The connected components of $\widehat{X}^{\mathrm{an}} \setminus V$ are the formal fibers of $\mathfrak{X}$, so it follows from the anti-continuity of red that $U = \mathrm{red}^{-1}(\overline{\mathfrak{C}})$ and that $\pi_0(U \setminus \{x\}) \xrightarrow{\sim} \overline{\mathfrak{C}}(k)$. To summarize:

LEMMA. *A simple neighborhood U of a type-2 point $x \in X^{\mathrm{an}}$ is the inverse image of a smooth irreducible component $\overline{\mathfrak{C}}$ of the special fiber of a semistable formal model $\mathfrak{X}$ of $\widehat{X}$. Furthermore, we have $\pi_0(U \setminus \{x\}) \xrightarrow{\sim} \overline{\mathfrak{C}}(k)$.*

5. The metric structure on an analytic curve

The set of all skeleta $\{\Sigma(X, V)\}_V$ is a filtered directed system under inclusion by Proposition 3.13. For U a one-dimensional K-analytic space, define the set of *skeletal points* $\mathbb{H}_\circ(U)$ of U to be the set of points of U of types 2 and 3, and the set of *norm points* to be $\mathbb{H}(U) := U \setminus U(K)$. When $U = X$ the latter are the points that arise from norms on the function field $K(X)$ which extend the given absolute value on K, and the following corollary explains the former terminology:

COROLLARY 5.1. *We have*

$$\mathbb{H}_\circ(X^{\mathrm{an}}) = \bigcup_V \Sigma(X, V) = \varinjlim_V \Sigma(X, V)$$

as sets, where V runs over all semistable vertex sets of X.

PROOF. Any point of $\Sigma(X, V)$ has type 2 or 3, and any type-2 point is contained in a semistable vertex set by Proposition 3.13(3). Let x be a type-3 point. Then x is contained in an open ball or an open annulus in a semistable decomposition of X^{an}. The semistable decomposition can then be refined as in the proof of Proposition 3.13(3) to produce a skeleton that includes x. $\square$

By Proposition 3.13(1), the set of all skeleta $\{\Sigma(X, V)\}_V$ is also an inverse system with respect to the natural retraction maps. Although not logically necessary for anything else in this paper, the following folklore counterpart to Corollary 5.1 is conceptually important. For a higher-dimensional analogue (without proof) in the case $\mathrm{char}(K) = 0$, see [**22**, Appendix A], and see [**11**, Corollary 3.2] in general. See also [**21**].

THEOREM 5.2. *The natural map*

$$u\colon \widehat{X}^{\mathrm{an}} \to \varprojlim_V \Sigma(\widehat{X}, V)$$

is a homeomorphism of topological spaces, where V runs over all semistable vertex sets of $\widehat{X}$.

PROOF. The map u exists and is continuous by the universal property of inverse limits. It is injective because given any two points $x \neq y$ in $\widehat{X}^{\mathrm{an}}$, one sees easily that there is a semistable vertex set V such that x and y retract to different points of $\Sigma(\widehat{X}, V)$. Since $\widehat{X}^{\mathrm{an}}$ is compact and each individual retraction map $\widehat{X}^{\mathrm{an}} \to \Sigma(\widehat{X}, V)$ is continuous and surjective, it follows from [**12**, §9.6, Corollary 2] that u is also surjective. By Proposition 8 in §9.6 of loc. cit., the space $\varprojlim_V \Sigma(\widehat{X}, V)$ is compact. Therefore u is a continuous bijection between compact (Hausdorff) spaces, hence a homeomorphism (cf. Corollary 2 in §9.4 of loc. cit.). □

5.3. The metric structure on $\mathbb{H}_\circ(X^{\mathrm{an}})$. Let $V \subset V'$ be semistable vertex sets of X. By Proposition 3.13(3) every edge e of $\Sigma(X, V)$ includes isometrically into an edge of $\Sigma(X, V')$. Let $x, y \in \Sigma(X, V)$ and let $[x, y]$ be a shortest path from x to y in $\Sigma(X, V)$. Then $[x, y]$ is also a shortest path in $\Sigma(X, V')$: if there were a shorter path $[x, y]'$ in $\Sigma(X, V')$ then $[x, y] \cup [x, y]'$ would represent a homology class in $H_1(\Sigma(X, V'), \mathbb{Z})$ that did not exist in $H_1(\Sigma(X, V), \mathbb{Z})$, which is impossible by the genus formula (4.1). Therefore the inclusion $\Sigma(X, V) \hookrightarrow \Sigma(X, V')$ is an isometry (with respect to the shortest-path metrics), so by Corollary 5.1 we obtain a natural metric ρ on $\mathbb{H}_\circ(X^{\mathrm{an}})$, called the *skeletal metric*.

Let V be a semistable vertex set and let $\tau = \tau_V\colon X^{\mathrm{an}} \to \Sigma(X, V)$ be the retraction onto the skeleton. If $x, y \in \mathbb{H}_\circ(X^{\mathrm{an}})$ are not contained in the same connected component of $X^{\mathrm{an}} \setminus \Sigma(X, V)$ then a shortest path from x to y in a larger skeleton must go through $\Sigma(X, V)$. It follows that

$$\rho(x, y) = \rho\big(x, \tau(x)\big) + \rho\big(\tau(x), \tau(y)\big) + \rho(\tau(y), y). \tag{5.1}$$

REMARK 5.4. (1) By definition any skeleton includes isometrically into $\mathbb{H}_\circ(X^{\mathrm{an}})$.

(2) It is important to note that the metric topology on $\mathbb{H}_\circ(X^{\mathrm{an}})$ is *stronger* than the subspace topology.

We can describe the skeletal metric locally as follows. By Berkovich's classification theorem, any point $x \in \mathbb{H}(\mathbb{A}^{1,\mathrm{an}})$ is a limit of Gauss points of balls of radii r_i converging to $r \in (0, \infty)$. We define $\mathrm{diam}(x) = r$. Any two points $x \neq y \in \mathbb{A}^{1,\mathrm{an}}$ are contained in a unique smallest closed ball; its Gauss point is denoted $x \vee y$. For $x, y \in \mathbb{H}(\mathbb{A}^{1,\mathrm{an}})$ we define

$$\rho_p(x, y) = 2\log\big(\mathrm{diam}(x \vee y)\big) - \log\big(\mathrm{diam}(x)\big) - \log\big(\mathrm{diam}(y)\big).$$

Then ρ_p is a metric on $\mathbb{H}(\mathbb{A}^{1,\mathrm{an}})$, called the *path distance metric*; see [**3**, §2.7]. If A is a standard open ball or standard generalized open annulus then the restriction of ρ_p to $\mathbb{H}(A)$ is called the *path distance metric* on $\mathbb{H}(A)$.

PROPOSITION 5.5. *Let $A \subset X^{\mathrm{an}}$ be an analytic domain isomorphic to a standard open ball or a standard generalized open annulus. Then the skeletal metric on $\mathbb{H}_\circ(X^{\mathrm{an}})$ and the path distance metric on $\mathbb{H}(A)$ restrict to the same metric on $\mathbb{H}_\circ(A)$.*

PROOF. Let V be a semistable vertex set containing the limit boundary of A (cf. Remark 4.13). Then $V \setminus (V \cap A)$ is a semistable vertex set since the connected components of $A \setminus (V \cap A)$ are connected components of $X^{\mathrm{an}} \setminus V$. Hence we may and do assume that A is a connected component of $X^{\mathrm{an}} \setminus V$. Suppose that A is an open ball, and fix an isomorphism $A \cong \mathbb{B}(a)_+$. Let $x, y \in A$ be type-2 points.

(1) Suppose that $x \vee y \in \{x, y\}$; without loss of generality we may assume that $x = x \vee y$. After recentering, we may assume in addition that x is the Gauss point of $\mathbb{B}(b)$ and that y is the Gauss point of $\mathbb{B}(c)$. Then the standard open annulus $A' = \mathbb{B}(b)_+ \setminus \mathbb{B}(c)$ is a connected component of $A \setminus \{x, y\}$, which breaks up into a disjoint union of open balls and the open annuli A' and $\mathbb{B}(a)_+ \setminus \mathbb{B}(b)$. Hence $V \cup \{x, y\}$ is a semistable vertex set, and $\Sigma(A')$ is the interior of the edge e of $\Sigma(X, V \cup \{x, y\})$ with endpoints x, y. Therefore $\rho(x, y)$ is the logarithmic modulus of A', which agrees with $\rho_p(x, y) = \log\bigl(\mathrm{diam}(x)\bigr) - \log\bigl(\mathrm{diam}(y)\bigr)$.

(2) Suppose that $z = x \vee y \notin \{x, y\}$. Then $A \setminus \{x, y, z\}$ is a disjoint union of open balls and three open annuli, two of which connect x, z and y, z. As above we have $\rho(x, y) = \rho(x, z) + \rho(y, z)$, which is the same as $\rho_p(x, y) = \bigl(\log\bigl(\mathrm{diam}(z)\bigr) - \log\bigl(\mathrm{diam}(x)\bigr)\bigr) + \bigl(\log\bigl(\mathrm{diam}(z)\bigr) - \log\bigl(\mathrm{diam}(y)\bigr)\bigr)$.

Since the type-2 points of A are dense [**3**, Lemma 1.8], this proves the claim when A is an open ball in a semistable decomposition of X. The proof when A is a generalized open annulus in a semistable decomposition of X has more cases but is not essentially any different, so it is left to the reader. □

Since Proposition 5.5 did not depend on the choice of isomorphism of A with a standard generalized open annulus, we obtain:

COROLLARY 5.6. *Any isomorphism of standard open balls or standard generalized open annuli induces an isometry with respect to the path distance metric.*

In particular, if A is an (abstract) open ball or generalized open annulus then we can speak of the path distance metric on $\mathbb{H}(A)$.

COROLLARY 5.7. *The metric ρ on $\mathbb{H}_\circ(X^{\mathrm{an}})$ extends in a unique way to a metric on $\mathbb{H}(X^{\mathrm{an}})$.*

PROOF. Let $x, y \in \mathbb{H}(X^{\mathrm{an}})$ and let V be a semistable vertex set of X. If x, y are contained in the same connected component $B \cong \mathbb{B}(1)_+$ of $X^{\mathrm{an}} \setminus \Sigma(X, V)$ then we set $\rho(x, y) = \rho_p(x, y)$. Otherwise we set

$$\rho(x, y) = \rho_p\bigl(x, \tau_V(x)\bigr) + \rho\bigl(\tau_V(x), \tau_V(y)\bigr) + \rho_p(\tau_V(y), y)$$

where we have extended the path distance metric ρ_p on a connected component B of $X^{\mathrm{an}} \setminus \Sigma(X)$ to its closure $B \cup \tau_V(B)$ by continuity (compare the proof of Lemma 3.2). By (5.1) and Proposition 5.5 this function extends ρ. We leave it to the reader to verify that ρ is a metric on $\mathbb{H}(X^{\mathrm{an}})$. □

5.8. A *geodesic segment from x to y* in a metric space T is the image of an isometric embedding $[a, b] \hookrightarrow T$ with $a \mapsto x$ and $b \mapsto y$. We often identify a geodesic segment with its image in T. Recall that an $\mathbb{R}$-*tree* is a metric space T with the following properties:

(1) For all $x, y \in T$ there is a unique geodesic segment $[x, y]$ from x to y.
(2) For all $x, y, z \in T$, if $[x, y] \cap [y, z] = \{y\}$ then $[x, z] = [x, y] \cup [y, z]$.

See [**3**, Appendix B]. It is proved in §1.4 of loc. cit. that $\mathbb{H}(\mathbb{B}(1))$ is an $\mathbb{R}$-tree under the path distance metric. It is clear that any path-connected subspace of an $\mathbb{R}$-tree is an $\mathbb{R}$-tree, so if A is an open ball or a generalized open annulus then $\mathbb{H}(A)$ is an $\mathbb{R}$-tree as well.

PROPOSITION 5.9. *Every point $x \in \mathbb{H}(X^{\mathrm{an}})$ admits a fundamental system of simple neighborhoods $\{U_\alpha\}$ in X^{an} such that $U_\alpha \cap \mathbb{H}(X^{\mathrm{an}})$ is an $\mathbb{R}$-tree under the restriction of ρ.*

The definition of a simple neighborhood of a point $x \in X^{\mathrm{an}}$ is found in Definition 4.28.

PROOF. If x has type 3 or 4 then a simple neighborhood of x is an open ball or an open annulus, so the proposition follows from Corollary 4.27 and Proposition 5.5. Let x be a type-2 point and let V be a semistable vertex set of X containing x. For small enough $\epsilon > 0$ the set $W = \{y \in \Sigma(X, V) : \rho(x, y) < \epsilon\}$ is simply-connected; fix such an $\epsilon \in G$, and let $U = \tau_V^{-1}(W)$. Then U is a simple neighborhood of x. We claim that $\mathbb{H}(U)$ is an $\mathbb{R}$-tree. Any connected component A of $U \setminus \{x\}$ is an open ball or an open annulus, so $\mathbb{H}(A)$ is an $\mathbb{R}$-tree. Moreover $\mathbb{H}(A) \cup \{x\}$ is isometric to a path-connected subspace of $\mathbb{H}\big(\mathbb{B}(1)\big)$ as in the proof of Lemma 3.2; it follows that $\mathbb{H}(A) \cup \{x\}$ is an $\mathbb{R}$-tree. Therefore $\mathbb{H}(U)$ is a collection of $\mathbb{R}$-trees joined together at the single point x, and the hypotheses on W along with (5.1) imply that if $y, z \in \mathbb{H}(U)$ are contained in different components of $U \setminus \{x\}$ then $\rho(y, z) = \rho(y, x) + \rho(x, z)$. It is clear that such an object is again an $\mathbb{R}$-tree. □

COROLLARY 5.10. *Let $x, y \in \mathbb{H}_\circ(X^{\mathrm{an}})$ and let $\Sigma = \Sigma(X, V)$ be a skeleton containing x and y. Then any geodesic segment from x to y is contained in Σ.*

PROOF. Any path from x to y in Σ is by definition a geodesic segment. If x, y are contained in an open subset U such that $\mathbb{H}(U)$ is an $\mathbb{R}$-tree then the path from x to y in $\Sigma \cap U$ is the unique geodesic segment from x to y in $\mathbb{H}(U)$. The general case follows by covering a geodesic segment from x to y by (finitely many) such U. □

5.11. Tangent directions and the Slope formula. Let $x \in \mathbb{H}(X^{\mathrm{an}})$. A *nontrivial geodesic segment starting at x* is a geodesic segment $\alpha\colon [0, a] \hookrightarrow \mathbb{H}(X^{\mathrm{an}})$ with $a > 0$ such that $\alpha(0) = x$. We say that two nontrivial geodesic segments α, α' starting at x are *equivalent at x* if α and α' agree on a neighborhood of 0. Following [**3**, §B.6], we define the set of *tangent directions at x* to be the set T_x of nontrivial geodesic segments starting at x up to equivalence at x. It is clear that T_x only depends on a neighborhood of x in X^{an}.

LEMMA 5.12. *Let $x \in \mathbb{H}(X^{\mathrm{an}})$ and let U be a simple neighborhood of x in X^{an}. Then $[x, y] \mapsto y$ establishes a bijection $T_x \xrightarrow{\sim} \pi_0(U \setminus \{x\})$. Moreover,*

(1) *If x has type 4 then there is only one tangent direction at x.*
(2) *If x has type 3 then there are two tangent directions at x.*
(3) *If x has type 2 then $U = \mathrm{red}^{-1}(\overline{\mathfrak{C}})$ for a smooth irreducible component $\overline{\mathfrak{C}}$ of the special fiber of a semistable formal model $\mathfrak{X}$ of $\widehat{X}$ by §4.29, and $T_x \xrightarrow{\sim} \pi_0(U \setminus \{x\}) \xrightarrow{\sim} \overline{\mathfrak{C}}(k)$.*

PROOF. We will assume for simplicity that $\mathbb{H}(U)$ is an $\mathbb{R}$-tree (i.e., that the induced metric on $\mathbb{H}(U)$ agrees with the shortest-path metric); the general case

reduces to this because U is contractible. The bijection $T_x \xrightarrow{\sim} \pi_0(\mathbb{H}(U) \setminus \{x\})$ is proved in [**3**, §B.6]. A connected component B of $U \setminus \{x\}$ is an $\mathbb{R}$-tree by Proposition 1.13 of loc. cit. and the type-1 points of B are leaves, so $\pi_0(\mathbb{H}(U) \setminus \{x\}) = \pi_0(U \setminus \{x\})$. Parts (1) and (2) are proved in §1.4 of loc. cit, and part (3) is §4.29. □

5.13. With the notation in Lemma 5.12(3), we have a canonical identification of $\widetilde{\mathscr{H}}(x)$ with the function field of $\overline{\mathfrak{C}}$ by [**4**, Proposition 2.4.4]. Hence we have an identification $\xi \mapsto \mathrm{ord}_\xi$ of $\overline{\mathfrak{C}}(k)$ with the set $\mathrm{DV}(\widetilde{\mathscr{H}}(x)/k)$ of nontrivial discrete valuations $\widetilde{\mathscr{H}}(x) \twoheadrightarrow \mathbb{Z}$ inducing the trivial valuation on k. One can prove that the composite bijection $T_x \xrightarrow{\sim} \mathrm{DV}(\widetilde{\mathscr{H}}(x)/k)$ is independent of the choice of U. The discrete valuation corresponding to a tangent direction $v \in T_x$ will be denoted $\mathrm{ord}_v \colon \widetilde{\mathscr{H}}(x) \to \mathbb{Z}$.

Let $x \in X^{\mathrm{an}}$ be a type-2 point and let f be an analytic function in a neighborhood of x. Let $c \in K^\times$ be a scalar such that $|f(x)| = c$. We define $\tilde{f}_x \in \widetilde{\mathscr{H}}(x)$ to be the residue of $c^{-1}f$, so $\tilde{f}_x$ is only defined up to multiplication by a nonzero scalar in k. However if $\mathrm{ord} \colon \widetilde{\mathscr{H}}(x) \to \mathbb{Z}$ is a nontrivial discrete valuation trivial on k then $\mathrm{ord}(\tilde{f}_x)$ is intrinsic to f.

Definition 5.14. A function $F \colon X^{\mathrm{an}} \to \mathbb{R}$ is *piecewise affine* provided that for any geodesic segment $\alpha \colon [a,b] \hookrightarrow \mathbb{H}(X^{\mathrm{an}})$ the pullback $F \circ \alpha \colon [a,b] \to \mathbb{R}$ is piecewise affine. The *outgoing slope* of a piecewise affine function F at a point $x \in \mathbb{H}(X^{\mathrm{an}})$ along a tangent direction $v \in T_x$ is defined to be

$$d_v F(x) = \lim_{\epsilon \to 0} (F \circ \alpha)'(\epsilon)$$

where $\alpha \colon [0,a] \hookrightarrow X^{\mathrm{an}}$ is a nontrivial geodesic segment starting at x which represents v. We say that a piecewise affine function F is *harmonic* at a point $x \in X^{\mathrm{an}}$ provided that the outgoing slope $d_v F(x)$ is nonzero for only finitely many $v \in T_x$, and $\sum_{v \in T_x} d_v F(x) = 0$. We say that F is *harmonic* if it is harmonic for all $x \in \mathbb{H}(X^{\mathrm{an}})$.

Theorem 5.15 (Slope Formula). *Let f be an algebraic function on X with no zeros or poles and let $F = -\log|f| \colon X^{\mathrm{an}} \to \mathbb{R}$. Let V be a semistable vertex set of X and let $\Sigma = \Sigma(X,V)$. Then:*

(1) *$F = F \circ \tau_\Sigma$ where $\tau_\Sigma \colon X^{\mathrm{an}} \to \Sigma$ is the retraction.*
(2) *F is piecewise affine with integer slopes, and F is affine-linear on each edge of Σ.*
(3) *If x is a type-2 point of X^{an} and $v \in T_x$ then $d_v F(x) = \mathrm{ord}_v(\tilde{f}_x)$.*
(4) *F is harmonic.*
(5) *Let $x \in D$, let e be the ray in Σ whose closure in $\widehat{X}$ contains x, let $y \in V$ be the other endpoint of e, and let $v \in T_y$ be the tangent direction represented by e. Then $d_v F(y) = \mathrm{ord}_x(f)$.*

Proof. The first claim follows from Lemma 2.13 and the fact that a unit on an open ball has constant absolute value. The linearity of F on edges of Σ is Proposition 2.10. Since $F = F \circ \tau_\Sigma$ we have that F is constant in a neighborhood of any point of type 4, and any geodesic segment contained in $\mathbb{H}_\circ(X^{\mathrm{an}})$ is contained in a skeleton by Corollary 5.10, so F is piecewise affine. The last claim is Proposition 2.10(2). The harmonicity of F is proved as follows: if $x \in X^{\mathrm{an}}$ has type 4 then

x has one tangent direction and F is locally constant in a neighborhood of x, so $\sum_{v\in T_x} d_v F(x) = 0$. If x has type 3 then x is contained in the interior of an edge e of a skeleton, and the two tangent directions v, w at x are represented by the two paths emanating from x in e; since F is affine on e we have $d_v F(x) = -d_w F(x)$. The harmonicity of F at type-2 points is an immediate consequence of (3) and the fact that the divisor of a meromorphic function on a smooth complete curve has degree zero.

The heart of this theorem is (3), which again is essentially a result of Bosch and Lütkebohmert. Let x be a type-2 point of X^{an}, let U be a simple neighborhood of x, and let $\mathfrak{X}$ be a semistable formal model of $\widehat{X}$ such that $x \in V(\mathfrak{X})$ and $U = \mathrm{red}^{-1}(\overline{\mathfrak{C}})$ where $\overline{\mathfrak{C}}$ is the smooth irreducible component of $\overline{\mathfrak{X}}$ with generic point $\mathrm{red}(x)$. We may and do assume that $V(\mathfrak{X})$ is a semistable vertex set of X containing V. Let $\overline{\mathfrak{C}}' \subset \overline{\mathfrak{C}}$ be the affine curve obtained by deleting all points $\xi \in \overline{\mathfrak{C}}$ which are not smooth in $\overline{\mathfrak{X}}$ and let $\mathfrak{C}'$ be the induced formal affine subscheme of $\mathfrak{X}$. Then $(\mathfrak{C}')^{\mathrm{an}} = \mathrm{red}^{-1}(\overline{\mathfrak{C}}') = \tau_{V(\mathfrak{X})}^{-1}(x)$ is an affinoid domain in X^{an} with Shilov boundary $\{x\}$. If we scale f such that $|f(x)| = 1$ then f and f^{-1} both have supremum norm 1 on $\tau_{V(\mathfrak{X})}^{-1}(x)$. It follows that the residue $\tilde{f}_x$ of f is a unit on $\overline{\mathfrak{C}}'$, so $\mathrm{ord}_\zeta(\tilde{f}_x) = 0$ for all $\zeta \in \overline{\mathfrak{C}}'(k)$. By (1) we have that F is constant on $\tau_{V(\mathfrak{X})}^{-1}(x)$, so $d_v F(x) = \mathrm{ord}_v(\tilde{f}_x) = 0$ for all $v \in T_x$ corresponding to closed points of $\overline{\mathfrak{C}}'$.

Now let $v \in T_x$ correspond to a point $\xi \in \overline{\mathfrak{C}}$ which is contained in two irreducible components $\overline{\mathfrak{C}}, \overline{\mathfrak{D}}$ of $\overline{\mathfrak{X}}$. Let $y \in X^{\mathrm{an}}$ be the point reducing to the generic point of $\overline{\mathfrak{D}}$ and let e be the edge in $\Sigma\big(X, V(\mathfrak{X})\big)$ connecting x and y, so e is a geodesic segment representing v. If e° is the interior of e then $A = \tau_{V(\mathfrak{X})}^{-1}(e^\circ) = \mathrm{red}^{-1}(\xi)$ in an open annulus; we let r be the modulus of A. By [**10**, Proposition 3.2] we have $F(x) - F(y) = -r \cdot \mathrm{ord}_\xi(\tilde{f}_x)$. Since F is affine on e we also have $F(x) - F(y) = -r \cdot d_v F(x)$, whence the desired equality. □

Remark 5.16. Theorem 5.15 is also proved in [**27**, Proposition 3.3.15], in the following form: if f is a nonzero meromorphic function on $\widehat{X}^{\mathrm{an}}$, then the extended real-valued function $\log|f|$ on X satisfies the differential equation

$$dd^c \log|f| = \delta_{\mathrm{div}(f)} \tag{5.2}$$

where dd^c is a distribution-valued operator which serves as a non-Archimedean analogue of the classical dd^c-operator on a Riemann surface. One can regard (5.2) as a non-Archimedean analogue of the classical 'Poincaré – Lelong formula' for Riemann surfaces. Since it would lead us too far astray to recall the general definition of Thuillier's dd^c-operator on an analytic curve, we simply call Theorem 5.15 the Slope Formula.

Remark 5.17. (1) See [**3**, Example 5.20] for a version of Theorem 5.15 for $X = \mathbb{P}^1$.

(2) It is an elementary exercise that conditions (4) and (5) of Theorem 5.15 uniquely determine the function $F\colon \Sigma \to \mathbb{R}$ up to addition by a constant; see the proof of [**3**, Proposition 3.2(A)].

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