EXTENSIONS OF TAUTOLOGICAL RINGS AND MOTIVIC STRUCTURES IN THE COHOMOLOGY OF $\overline{\mathcal{M}}_{g,n}$

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ABSTRACT. We study collections of subrings of $H^*(\overline{\mathcal{M}}_{g,n})$ that are closed under the tautological operations that map cohomology classes on moduli spaces of smaller dimension to those on moduli spaces of larger dimension and contain the tautological subrings. Such extensions of tautological rings are well-suited for inductive arguments and flexible enough for a wide range of applications. In particular, we confirm predictions of Chenevier and Lannes for the ℓ-adic Galois representations and Hodge structures that appear in $H^k(\overline{\mathcal{M}}_{g,n})$ for $k = 13$, 14, and 15. We also show that $H^4(\overline{\mathcal{M}}_{g,n})$ is generated by tautological classes for all g and n, confirming a prediction of Arbarello and Cornalba from the 1990s. In order to establish the final bases cases needed for the inductive proofs of our main results, we use Mukai's construction of canonically embedded pentagonal curves of genus 7 as linear sections of an orthogonal Grassmannian and a decomposition of the diagonal to show that the pure weight cohomolog of $\mathcal{M}_{7,n}$ is generated by algebraic cycle classes, for $n \leq 3$.

1. Introduction

The moduli spaces of stable curves $\overline{\mathcal{M}}_{g,n}$ are smooth and proper over the integers, and this implies strong restrictions on the motivic structures, such as ℓ-adic Galois representations, that can appear in $H^*(\overline{\mathcal{M}}_{g,n})$. Widely believed conjectures regarding analytic continuations and functional equations for L-functions lead to precise predictions, by Chenevier and Lannes, about which such structures can appear in degrees less than or equal to 22 [10, Theorem F]. These predictions are consistent with all previously known results on $H^*(\overline{\mathcal{M}}_{g,n})$. Recent work inspired by these predictions confirms their correctness in all degrees less than or equal to 12 [4, 8]. Here we introduce new methods to sytematically study the motivic structures in $H^k(\overline{\mathcal{M}}_{g,n})$ for $k > 12$ and confirm these predictions in degrees 13, 14, and 15.

Throughout, we write $H^*(X)$ for the rational singular cohomology of a scheme or Deligne-Mumford stack X endowed with its associated Hodge structure or ℓ-adic Galois representation and $H^*(X)^{\mathrm{ss}}$ for its semisimplification. Let $\mathsf{L} := H^2(\mathbb{P}^1)$ and $\mathsf{S}_{12} := H^{11}(\overline{\mathcal{M}}_{1,11})$.

Theorem 1.1. *For all g and n, we have $H^{13}(\overline{\mathcal{M}}_{g,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{L}\mathsf{S}_{12}$ and $H^{14}(\overline{\mathcal{M}}_{g,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{L}^7$. Moreover, for $g \geq 2$, we have $H^{15}(\overline{\mathcal{M}}_{g,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{L}^2\mathsf{S}_{12}$.*

Theorem 1.1 confirms the predictions of Chenevier and Lannes for motivic weights $k \leq 15$. Note that the Hodge structure on the cohomology of a smooth and proper Deligne–Mumford stack such as $\overline{\mathcal{M}}_{g,n}$ is semi-simple, so the semi-simplification in Theorem 1.1 is relevant only when considering ℓ-adic Galois representations.

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The proof of Theorem 1.1 uses the inductive structure of the boundary of the moduli space and the maps induced by tautological morphisms between moduli spaces, as do the proofs of the precursor results mentioned above. Recall that the collection of tautological rings $RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ is the smallest collection of subrings that is closed under pushforward and pullback for the tautological morphisms induced by gluing, forgetting, and permuting marked points. For many inductive arguments, including those used here, it suffices to consider the operations that produce cohomology classes on moduli spaces of larger dimension from those on moduli spaces of smaller dimension.

Definition 1.2. A *semi-tautological extension* (STE) is a collection of subrings $S^*(\overline{\mathcal{M}}_{g,n})$ of $H^*(\overline{\mathcal{M}}_{g,n})$ that contains the tautological subrings $RH^*(\overline{\mathcal{M}}_{g,n})$ and is closed under pullback by forgetting and permuting marked points and under pushforward for gluing marked points.

Examples of STEs include the trivial extension RH^*, the full cohomology rings H^*, and the collection of subrings generated by algebraic cycle classes. Not every STE is closed under the additional tautological operations induced by push-forward for forgetting marked points and pullback for gluing marked points. However, the main examples we study here are indeed closed under all of the tautological operations. See Proposition 2.3.

Note that any intersection of STEs is an STE. An STE is *finitely generated* if it is the smallest STE that contains a given finite subset (or, equivalently, the union of finitely many $\mathbb{Q}$-vector subspaces) of $\coprod_{g,n} H^*(\overline{\mathcal{M}}_{g,n})$. A finitely generated STE is suitable for combinatorial study via algebraic operations on decorated graphs. See [20, 28] for discussions of the graphical algebra underlying the tautological ring, and [22, 23] for applications of such operadic methods, with not necessarily tautological decorations, to the weight spectral sequence for $\mathcal{M}_{g,n}$. Every STE that we consider is *motivic*, meaning that $S^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ is a sub-Hodge structure and its base change to $\mathbb{Q}_\ell$ is preserved by the Galois action.

Theorem 1.3. *For any fixed degree k, the STE generated by*

$$\left\{ H^{k'}(\overline{\mathcal{M}}_{g',n'}) : k' \leq k, g' < \tfrac{3}{2}k' + 1,\ n' \leq k',\ 4g' - 4 + n' \geq k' \right\}$$

contains $H^k(\overline{\mathcal{M}}_{g,n})$ for all g and n.

In particular, there is a finitely generated STE that contains $H^k(\overline{\mathcal{M}}_{g,n})$ for all g and n.

Corollary 1.4. *For each k, there are only finitely many isomorphism classes of simple Hodge structures (resp. ℓ-adic Galois representations) in $\bigoplus_{g,n} H^k(\overline{\mathcal{M}}_{g,n})^{\mathrm{ss}}$.*

We developed the notion of STEs to study nontrivial extensions of tautological rings, such as the STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$, but the same methods also yield new results on the tautological ring itself. We apply the explicit bounds in Theorem 1.3, together with new tools and results for small g, n, and k to prove the following. At the level of $\mathbb{Q}$-vector spaces, we identify $H_k(\overline{\mathcal{M}}_{g,n})$ with $H^{2d_{g,n}-k}(\overline{\mathcal{M}}_{g,n})$, where $d_{g,n} := 3g-3+n$ is the dimension of $\overline{\mathcal{M}}_{g,n}$. Similarly, when S^* is an STE we write $S_k(\overline{\mathcal{M}}_{g,n})$ for the $\mathbb{Q}$-vector space $S^{2d_{g,n}-k}(\overline{\mathcal{M}}_{g,n})$.

Theorem 1.5. *The tautological ring $RH^*(\overline{\mathcal{M}}_{g,n})$ contains*

(1) $H^4(\overline{\mathcal{M}}_{g,n})$, for all g and n,
(2) $H^6(\overline{\mathcal{M}}_{g,n})$, for $g \geq 10$,
(3) $H_k(\overline{\mathcal{M}}_{g,n})$, for even $k \leq 14$, for all g and n.

Theorem 1.6. *The STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$ contains $H_{13}(\overline{\mathcal{M}}_{g,n})$ for all g and n.*

Theorem 1.7. *The STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$, $H^{15}(\overline{\mathcal{M}}_{1,15})$, $H^{15}(\overline{\mathcal{M}}_{2,12})$ and $H^{15}(\overline{\mathcal{M}}_{2,13})$ contains $H_{15}(\overline{\mathcal{M}}_{g,n})$ for all g and n.*

Theorem 1.5(1) confirms a prediction of Arbarello and Cornalba from the 1990s; they proposed that their inductive method used to prove that $H^2(\overline{\mathcal{M}}_{g,n})$ is tautological should also apply in degree 4 [1, p. 1]. Shortly thereafter, Polito confirmed that $H^4(\overline{\mathcal{M}}_{g,n})$ is tautological for $g \geq 8$ [29], but the general case remained open until now.

Theorem 1.5(3) implies that $H^k(\overline{\mathcal{M}}_{g,n}) \cong \bigoplus \mathsf{L}^{k/2}$ for even $k \leq 14$. The work of Chenevier and Lannes predicts that $H^k(\overline{\mathcal{M}}_{g,n})$ should also be isomorphic to $\bigoplus \mathsf{L}^{k/2}$ for $k = 16, 18, 20$. The Hodge and Tate conjectures then predict that these groups are generated by algebraic cycle classes. However, generation by algebraic cycle classes is an open problem except in the cases where $H^k(\overline{\mathcal{M}}_{g,n})$ is known to be generated by tautological cycle classes.

We note that any STE is top-heavy, in the sense that $\dim S^k(\overline{\mathcal{M}}_{g,n}) \leq \dim S_k(\overline{\mathcal{M}}_{g,n})$ for $k \leq \dim \overline{\mathcal{M}}_{g,n}$, cf. [27]. Moreover, the dimensions in even and odd degrees are unimodal. This is because any STE contains RH^* and hence contains an ample class. Multiplication by a suitable power of the ample class gives an injection from $S^k(\overline{\mathcal{M}}_{g,n})$ to $S_k(\overline{\mathcal{M}}_{g,n})$.

One could show that $H^k(\overline{\mathcal{M}}_{g,n})$ is generated by tautological cycles in the remaining cases covered by Theorem 1.5(3) by showing that the pairing on $RH^k(\overline{\mathcal{M}}_{g,n}) \times RH_k(\overline{\mathcal{M}}_{g,n})$ is perfect, e.g., using admcycles [11]. However, computational complexity prevents meaningful progress by brute force. Note that Petersen and Tommasi showed that this pairing is not perfect in general for $k \geq 22$ [26,27]. Graber and Pandharipande had previously shown that $H^{22}(\overline{\mathcal{M}}_{2,20})$ contains an algebraic cycle class that is not tautological [15].

Conjecture 1.8. *The tautological ring $RH^*(\overline{\mathcal{M}}_{g,n})$ contains $H^k(\overline{\mathcal{M}}_{g,n})$ for even $k \leq 20$.*

The results above show that Conjecture 1.8 is true for $k \leq 4$, and for $k = 6$ and $g \geq 10$. For $k \leq 14$, the conjecture is true if and only if the pairing $RH^k(\overline{\mathcal{M}}_{g,n}) \times RH_k(\overline{\mathcal{M}}_{g,n})$ is perfect. The examples of Graber–Pandharipande and Petersen–Tommasi show that the conjectured bound of $k \leq 20$ is the best possible. As further evidence for Conjecture 1.8, we note that the Arbarello–Cornalba induction together with known base cases implies the vanishing of $H^{16,0}(\overline{\mathcal{M}}_{g,n})$ and $H^{18,0}(\overline{\mathcal{M}}_{g,n})$ for all g and n, as recently observed by Fontanari [13].

The inductive arguments used to study $H^k(\overline{\mathcal{M}}_{g,n})$ for all g and n rely on understanding base cases $H^{k'}(\overline{\mathcal{M}}_{g',n'})$ where g' and n' are small relative to k. As k grows, more base cases are needed. With the exception of $H^{11}(\overline{\mathcal{M}}_{1,11})$, all base cases required for previous work with $k \leq 12$ have been pure Hodge–Tate [1,4,5]. Substantial work went into establishing these base cases via point counts and other methods. When $k \geq 13$, the problem becomes fundamentally more difficult, as an increasing number of the required base cases are *not* pure Hodge–Tate. In particular, previous techniques for handling base cases do not apply.

The advances presented here depend on a new technique for controlling the pure weight cohomology of $\mathcal{M}_{g,n}$. A space X has the *Chow–Künneth generation Property (CKgP)* if the tensor product map on Chow groups $A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$ is surjective for all Y. If X is smooth, proper, and has the CKgP, then the cycle class map is an isomorphism. However, in several of the base cases needed for our arguments, the smooth and proper moduli space $\overline{\mathcal{M}}_{g,n}$ has odd cohomology and hence does not have the CKgP. Nevertheless, we show that the open moduli spaces $\mathcal{M}_{g,n}$ do have the CKgP for the relevant pairs (g, n). In order to apply this in the proof of our main results, the key new technical statement is Lemma 4.3, which says that if X is smooth and has the CKgP, then $W_k H^k(X)$ is algebraic. This extension of the aforementioned result on the cycle class map for smooth and proper spaces with the CKgP to spaces that are not necessarily proper is essential for our the purpose of controlling the motivic structures that appear in $H^k(\overline{\mathcal{M}}_{g,n})$ for $k \geq 13$.

For example, using that $\mathcal{M}_{3,n}$ has the CKgP for $n \leq 11$ [6, Theorem 1.4], we determine the Hodge structures and Galois representations that appear in $H^*(\overline{\mathcal{M}}_{3,n})$, for $n \leq 11$.

Theorem 1.9. *For $n \leq 11$, $H^*(\overline{\mathcal{M}}_{3,n})^{\mathrm{ss}}$ is a polynomial in L and S_{12}.*

Bergström and Faber recently used point counting techniques to compute the cohomology of $\overline{\mathcal{M}}_{3,n}$ as an $\mathbb{S}_n$-equivariant Galois representation for $n \leq 14$. For $n \geq 9$, these computations are conditional on the assumption that the only ℓ-adic Galois representations appearing are those from the list of Chenevier and Lannes [3]. Theorem 1.9 unconditionally confirms the calculations of Bergström and Faber for $n = 9$, 10, and 11.

In order to prove Theorems 1.1, 1.5(3), and 1.7, our inductive arguments require several base cases beyond what was already in the literature. In particular, we prove the following results in genus 7, which are also of independent interest.

Theorem 1.10. *For $n \leq 3$, the moduli space $\mathcal{M}_{7,n}$ has the CKgP and $R^*(\mathcal{M}_{7,n}) = A^*(\mathcal{M}_{7,n})$.*

Here, R^* denotes the subring of the Chow ring A^* generated by tautological cycle classes. Previous results proving that $\mathcal{M}_{g,n}$ has the CKgP and $R^*(\mathcal{M}_{g,n}) = A^*(\mathcal{M}_{g,n})$ for small g and n have primarily relied on corresponding results for Hurwitz spaces with marked points [6]. Unfortunately, the numerics for degree 5 covers prevented this technique from working with marked points. Here, we take a new approach to the pentagonal locus, using a modification of Mukai's construction [19] that includes markings.

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2. Preliminaries

2.1. **Preliminaries.** In this section, we establish notation and terminology that we will use throughout the paper and discuss a few basic examples of STEs. We also recall previously known facts about the cohomology groups of moduli spaces, especially in genus 0, 1, and 2, that will be used in the base cases of our inductive arguments.

Recall that an STE is, by definition, closed under the tautological operations given by pushing forward from the boundary or pulling back from moduli spaces with fewer marked points. Let $\widetilde{\partial \mathcal{M}}_{g,n}$ denote the normalization of the boundary. The sequence

$$H^{k-2}(\widetilde{\partial \mathcal{M}}_{g,n}) \to H^k(\overline{\mathcal{M}}_{g,n}) \to W_k H^k(\mathcal{M}_{g,n}) \to 0$$

is right exact.

Let $\pi_i \colon \mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ be the tautological morphism forgetting the ith marking and let

$$\Phi^k_{g,n} := \pi_1^* W_k H^k(\mathcal{M}_{g,n-1}) + \ldots + \pi_n^* W_k H^k(\mathcal{M}_{g,n-1}) \subset W_k H^k(\mathcal{M}_{g,n}).$$

The following lemma is a cohomological analogue of the "filling criteria" in [6, Section 4].

We consider the partial order in which $(g',n') \prec (g,n)$ if $g' \leq g$ and $2g'+n' \leq 2g+n$ and $(g',n') \neq (g,n)$. The moduli space $\overline{\mathcal{M}}_{g,n}$ is stratified according to the topological types of stable curves, and each stratum of the boundary is a finite quotient of a product of moduli spaces $\mathcal{M}_{g',n'}$ such that $(g',n') \prec (g,n)$. Recall that we write $d_{g,n} := 3g-3+n$.

Lemma 2.1. *Let S^* be an STE and let $2g-2+n>0$. If the canonical map*

$$S^{k'}(\overline{\mathcal{M}}_{g',n'}) \to W_{k'}H^{k'}(\mathcal{M}_{g',n'})/\big(\Phi^{k'}_{g',n'} + RH^{k'}(\mathcal{M}_{g',n'})\big) \tag{2.1}$$

is surjective for $(g',n',k') = (g,n,k)$ and all (g',n',k') satisfying

$$(g',n') \prec (g,n), \qquad 2d_{g',n'} - k' \leq 2d_{g,n} - k \qquad \text{and} \qquad k' \leq k-2 \tag{2.2}$$

then $S^k(\overline{\mathcal{M}}_{g,n}) = H^k(\overline{\mathcal{M}}_{g,n})$.

Proof. The proof is by induction on g and n. Consider the diagram

$$\begin{tikzcd}
S^{k-2}(\widetilde{\partial\mathcal{M}_{g,n}}) \arrow[r] \arrow[d, "\alpha"] & S^k(\overline{\mathcal{M}}_{g,n}) \arrow[d, "\beta"] \arrow[dr, "\phi"] & & \\
H^{k-2}(\widetilde{\partial\mathcal{M}_{g,n}}) \arrow[r] & H^k(\overline{\mathcal{M}}_{g,n}) \arrow[r] & W_kH^k(\mathcal{M}_{g,n}) \arrow[r] & 0.
\end{tikzcd} \tag{2.3}$$

Here, we extend S^* to $\widetilde{\partial\mathcal{M}_{g,n}}$ in the natural way, by summing over components, using the Künneth formula, and taking invariants under automorphisms of the dual graph. We claim that α is an isomorphism. By the Künneth formula, the domain of α is a sum of tensor products of $S^\ell(\overline{\mathcal{M}}_{\gamma,\nu})$ with $\ell \leq k-2$ and $(\gamma,\nu) \prec (g,n)$. Furthermore, by considering dimensions of the cycles involved, we must also have $2d_{\gamma,\nu} - \ell \leq 2d_{g,n} - k$. This is because we must have $k-2-\ell \leq 2(d_{g,n} - 1 - d_{\gamma,\nu})$, so that the degree of the other Künneth component does not exceed its real dimension. Now, suppose we are given (g',n',k') that satisfy

$$(g',n') \prec (\gamma,\nu), \qquad 2d_{g',n'} - k' \leq 2d_{\gamma,\nu} - \ell \qquad \text{and} \qquad k' \leq \ell - 2.$$

Then (g',n',k') satisfies (2.2), so $S^\ell(\overline{\mathcal{M}}_{\gamma,\nu}) = H^\ell(\overline{\mathcal{M}}_{\gamma,\nu})$ by induction. This proves the claim.

Next, by induction we have $H^k(\overline{\mathcal{M}}_{g,n-1}) = S^k(\overline{\mathcal{M}}_{g,n-1})$, so the image of

$$\pi_i^* \colon H^k(\overline{\mathcal{M}}_{g,n-1}) \to H^k(\overline{\mathcal{M}}_{g,n})$$

is contained in $S^k(\overline{\mathcal{M}}_{g,n})$ for all i. Hence, $\Phi^k_{g,n}$ is contained in the image of ϕ. The tautological classes $RH^k(\mathcal{M}_{g,n}) \subseteq W_kH^k(\mathcal{M}_{g,n})$ are also contained in the image of ϕ by definition. Thus, the surjectivity of

$$S^k(\overline{\mathcal{M}}_{g,n}) \to W_kH^k(\mathcal{M}_{g,n})/\left(\Phi^k_{g,n} + RH^k(\mathcal{M}_{g,n})\right)$$

implies that ϕ is surjective. Hence, β is also surjective, as desired. $\square$

In order to apply Lemma 2.1, we need results that help us understand generators for $W_kH^k(\mathcal{M}_{g,n})/\left(\Phi^k_{g,n} + RH^k(\mathcal{M}_{g,n})\right)$. This is the topic of Sections 3 and 4.

2.2. **Pure cohomology in genus 1 and 2.** On $\mathcal{M}_{0,n}$, the only nonvanishing pure cohomology is $W_0H^0(\mathcal{M}_{0,n})$. Below, we present the complete classification of pure cohomology in genus 1, which is due to Getzler [14]. In genus 2, we present a classification in low cohomological degree, following Petersen [25].

2.2.1. *Genus 1.* The following statement appeared in [21, Proposition 7]; the proof there is omitted. Here, we include the proof (and corrected statement), which was explained to us by Dan Petersen. Let S_{k+1} denote the weight k structure associated to the space of cusp forms of weight $k+1$ for $\mathrm{SL}_2(\mathbb{Z})$. By the Eichler–Shimura correspondence, we have $\mathsf{S}_{k+1} = W_k H^1(\mathcal{M}_{1,1}, \mathbb{V}^{\otimes k-1})$. We will see that the latter is identified with $W_k H^k(\mathcal{M}_{1,k})$. More generally, we have the following. Given a partition λ of n, let V_λ be the associated Specht module representation of $\mathbb{S}_n$.

Proposition 2.2. *For $n \geq k$, a basis for $W_k H^k(\mathcal{M}_{1,n})$ is given by the $\binom{n-1}{k-1}$ pullbacks from $W_k H^k(\mathcal{M}_{1,A})$ where A runs over all subsets of $\{1, \dots, n\}$ of size k such that $1 \in A$. Consequently, there is an $\mathbb{S}_n$-equivariant isomorphism*

$$W_k H^k(\mathcal{M}_{1,n}) \cong \mathsf{S}_{k+1} \otimes V_{n-k+1,1^{k-1}}.$$

For $n < k$, we have $W_k H^k(\mathcal{M}_{1,n}) = 0$.

Proof. Let $\pi\colon E \to \mathcal{M}_{1,1}$ denote the universal elliptic curve, and $\sigma\colon \mathcal{M}_{1,1} \to E$ the section. Associated to the open embedding $\mathcal{M}_{1,n} \hookrightarrow E^{n-1}$, we have a right exact sequence

$$\bigoplus W_{k-2} H^{k-2}(E^{n-2}) \to W_k H^k(E^{n-1}) \to W_k H^k(\mathcal{M}_{1,n}) \to 0. \tag{2.4}$$

Since $f\colon E^{n-1} \to \mathcal{M}_{1,1}$ is smooth and proper, using the Leray spectral sequence, we have

$$H^k(E^{n-1}) = \bigoplus_{p+q=k} H^p(\mathcal{M}_{1,1}, R^q f_* \mathbb{Q}). \tag{2.5}$$

By the Künneth formula, we have

$$R^q f_* \mathbb{Q} = \bigoplus_{i_2 + \ldots + i_n = q} R^{i_2}\pi_*\mathbb{Q} \otimes \cdots \otimes R^{i_n}\pi_*\mathbb{Q}.$$

Let $\mathbb{V} := R^1\pi_*\mathbb{Q}$ and note that $R^0\pi_*\mathbb{Q} = \mathbb{Q}$ and $R^2\pi_*\mathbb{Q} = \mathbb{Q}(-1)$.

Fix some p and $(i_2, \dots, i_n)$ with $p + i_2 + \ldots + i_n = k$. If $i_j = 2$, then we claim that

$$W_k H^p(\mathcal{M}_{1,1}, R^{i_2}\pi_*\mathbb{Q} \otimes \cdots \otimes R^{i_n}\pi_*\mathbb{Q}) \subset W_k H^k(E^{n-1}) \tag{2.6}$$

lies in the image of of the first map in (2.4). More precisely, let $\alpha_j\colon E^{n-2} \to E^{n-1}$, be the locus where the jth entry in E^{n-1} agrees with the section $\sigma\colon \mathcal{M}_{1,1} \to E$. In other words, α_j is defined by the fiber diagram

$$\begin{array}{ccc} E^{n-2} & \xrightarrow{\alpha_j} & E^{n-1} \\ \big\downarrow & & \big\downarrow \scriptstyle{\mathrm{pr}_j} \\ \mathcal{M}_{1,1} & \xrightarrow[\sigma]{} & E. \end{array}$$

If $i_j = 2$, we have $p + i_2 + \ldots + i_{j-1} + i_{j+1} + \ldots + i_n = k-2$. Therefore, using the Leray spectral sequence for $E^{n-2} \to \mathcal{M}_{1,1}$, there is a corresponding term

$$W_k H^p(\mathcal{M}_{1,1}, R^{i_2}\pi_*\mathbb{Q} \otimes \cdots \otimes R^{i_{j-1}}\pi_*\mathbb{Q} \otimes R^{i_{j+1}}\pi_*\mathbb{Q} \otimes \cdots \otimes R^{i_n}\pi_*\mathbb{Q}) \subset W_k H^{k-2}(E^{n-2}). \tag{2.7}$$

Then the pushforward $\alpha_{j*}\colon H^{k-2}(E^{n-2}) \to H^k(E^{n-1})$ sends the subspace on the left of (2.7) isomorphically onto the subspace on the left of (2.6), which proves the claim

It follows that $W_kH^k(\mathcal{M}_{1,n})$ is generated by the terms $W_kH^p(\mathcal{M}_{1,1}, R^{i_2}\pi_*\mathbb{Q}\otimes\cdots\otimes R^{i_n}\pi_*\mathbb{Q})$ in $W_kH^k(E^{n-1})$ where all $i_j \leq 1$. By [24, Section 2], we have $W_kH^p(\mathcal{M}_{1,1}, \mathbb{V}^{\otimes q}) = 0$ unless $p = 1$ and $q = k-1$, in which case $W_kH^1(\mathcal{M}_{1,1}, \mathbb{V}^{\otimes k-1}) = \mathsf{S}_{k+1}$ by Eichler–Shimura.

There are $\binom{n-1}{k-1}$ terms of the form $H^1(\mathcal{M}_{1,1}, \mathbb{V}^{\otimes k-1})$ in (2.5), coming from choosing which $k-1$ of the $n-1$ indices have $i_j = 1$. Each of these terms is pulled back along the projection map $E^{n-1} \to E^{k-1}$, which remembers the $k-1$ factors for which $i_j = 1$. Let A be the collection of indices j such that $i_j = 1$ together with 1. There is a commutative diagram

$$\begin{array}{ccc} W_kH^k(E^{n-1}) & \longrightarrow & W_kH^k(\mathcal{M}_{1,n}) \\ \uparrow & & \uparrow \\ W_kH^k(E^{k-1}) & \longrightarrow & W_kH^k(\mathcal{M}_{1,A}). \end{array}$$

It follows that $W_kH^k(\mathcal{M}_{1,n})$ is generated by the pullbacks from $W_kH^k(\mathcal{M}_{1,A})$ as A ranges over all subsets of size k containing 1. Finally, we note that there can be no relations among these $\binom{n-1}{k-1}$ copies of $W_kH^1(\mathcal{M}_{1,1}, \mathbb{V}^{\otimes k-1}) = \mathsf{S}_{k+1}$ since the image of the left-hand map of (2.4) lies in the subspace of type $\mathsf{L}^i\mathsf{S}_{k+1-2i}$ for $i \geq 1$.

We have now shown that $W_kH^k(\mathcal{M}_{1,n}) = \mathsf{S}_{k+1} \otimes U$ for some $\binom{n-1}{k-1}$-dimensional vector space U. From the discussion above, it is not difficult to identify U as an $\mathbb{S}_n$-representation. When $n = k$, the $\mathbb{S}_k$ action on $W_kH^k(\mathcal{M}_{1,k})$ is the sign representation. To identify U for $n > k$, let $\mathbb{S}_{n-1} \subset \mathbb{S}_n$ be the subgroup that fixes 1. Since $W_kH^k(\mathcal{M}_{1,n})$ is freely generated by the pullbacks from $W_kH^k(\mathcal{M}_{1,A})$ as A runs over subsets of size k containing 1, we have

$$\mathrm{Res}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} U = \mathrm{Ind}^{\mathbb{S}_{n-1}}_{\mathbb{S}_{k-1}\times\mathbb{S}_{n-k}}(\mathrm{sgn} \boxtimes \mathbf{1}).$$

By the Pieri rule, we have

$$\mathrm{Ind}^{\mathbb{S}_{n-1}}_{\mathbb{S}_{k-1}\times\mathbb{S}_{n-k}}(\mathrm{sgn} \boxtimes \mathbf{1}) = \mathrm{Ind}^{\mathbb{S}_{n-1}}_{\mathbb{S}_{k-1}\times\mathbb{S}_{n-k}}(V_{1^{k-1}} \boxtimes V_{n-k}) = V_{n-k+1,1^{k-2}} \oplus V_{n-k,1^{k-1}}. \tag{2.8}$$

By the branching rule, $V_{n-k+1,1^{k-1}}$ is the unique $\mathbb{S}_n$ representation whose restriction to $\mathbb{S}_{n-1}$ is the representation in (2.8). $\square$

Let S^*_ω denote the STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$. Note that S^*_ω is the STE generated by the class $\omega \in H^{11,0}(\overline{\mathcal{M}}_{1,11})$ associated to the weight 12 cusp form for $\mathrm{SL}_2(\mathbb{Z})$, i.e., it is the smallest STE whose complexification contains ω. In [8], we showed that S^*_ω contains $H^{11}(\overline{\mathcal{M}}_{g,n})$, and hence $H_{11}(\overline{\mathcal{M}}_{g,n})$, for all g and n. In this paper, we show that it also contains $H_{13}(\overline{\mathcal{M}}_{g,n})$ for all g and n (Theorem 1.6). In contrast with the system of tautological rings, an arbitrary STE need not be closed under pushforward along maps forgetting marked points or pullback to the boundary. Nevertheless, we have the following result for $S^*_\omega(\overline{\mathcal{M}}_{g,n})$.

Proposition 2.3. *The STE S^*_ω is closed under the tautological operations induced by pushforward for forgetting marked points and pullback for gluing marked points.*

Proof. By [24], all even cohomology in genus 1 is represented by boundary strata. Therefore, any product of two odd degree classes can be written as a sum of boundary strata. It follows that every class in $S^*_\omega(\overline{\mathcal{M}}_{g,n})$ can be represented as a linear combination of decorated graphs all of whose nontautological decorations are H^{11}-classes on genus 1 vertices.

The pushforward of $H^{11}(\overline{\mathcal{M}}_{1,n})$ along the maps forgetting marked points is zero. Since the tautological rings are closed under pushforward, it follows that S^*_ω is closed under pushforward.

By [8, Lemma 2.2], the image of $H^{11}(\overline{\mathcal{M}}_{1,n})$ under pullback to the boundary lies in S^*_ω. By the excess intersection formula, and using the fact that ψ-classes in genus 1 are boundary classes, it follows that the pullback of any such decorated graph in S^*_ω to any boundary stratum is a sum of decorated graphs of the same form. In particular, S^*_ω is closed under pullback to the boundary, as required. $\square$

Remark 2.4. More generally, the Hodge groups $H^{k,0}(\overline{\mathcal{M}}_{1,k})$ correspond to the space of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight $k+1$. Essentially the same argument shows that the STE generated by any subset of these cusp form classes is closed under all of the tautological operations.

2.2.2. *Genus 2.* Here, we summarize what we need about the pure weight cohomology of $\mathcal{M}_{2,n}$ in low degrees, from [26, 27].

Proposition 2.5. *Let $k \leq 10$. Then $W_{2k}H^{2k}(\mathcal{M}_{2,n}) = RH^{2k}(\mathcal{M}_{2,n})$.*

Proof. When $n < 20$, we have $H^{2k}(\overline{\mathcal{M}}_{2,n}) = RH^{2k}(\overline{\mathcal{M}}_{2,n})$ by [26, Theorem 3.8]. Hence, $W_{2k}H^{2k}(\mathcal{M}_{2,n}) = RH^{2k}(\mathcal{M}_{2,n})$ by restriction. For $n = 20$ and $k \leq 10$, the same result holds (see [26, Remark 3.10]). For $n > 20$ and $k \leq 10$, all of $W_{2k}H^{2k}(\mathcal{M}_{2,n})$ is pulled back from $W_{2k}H^{2k}(\mathcal{M}_{2,m})$ where $m \leq 20$ (see Lemma 3.1, below). Because the tautological ring is closed under forgetful pullbacks, the lemma follows. $\square$

Applying Lemma 2.1 to the STE RH^* immediately implies Conjecture 1.8 for $g = 2$.

Corollary 2.6. *If $k \leq 10$, then $H^{2k}(\overline{\mathcal{M}}_{2,n}) = RH^{2k}(\overline{\mathcal{M}}_{2,n})$ for all n.*

In odd degrees, there are restrictions on the possible motivic structures coming from the cohomology of local systems on the moduli space of principally polarized abelian surfaces.

Proposition 2.7. *The pure weight cohomology of $\mathcal{M}_{2,n}$ in degrees 13 and 15 is of the form*

$$W_{13}H^{13}(\mathcal{M}_{2,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{LS}_{12} \quad \text{and} \quad W_{15}H^{15}(\mathcal{M}_{2,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{L}^2\mathsf{S}_{12}.$$

This follows by standard arguments with the Künneth formula, Leray spectral sequence, and the long exact sequence in cohomology induced by the Torelli map $\mathcal{M}_2 \to \mathcal{A}_2$ to reduce to Petersen's classification of the cohomology of local systems on $\mathcal{A}_2$ [25].

Corollary 2.8. *On $\overline{\mathcal{M}}_{2,n}$, we have*

$$H^{13}(\overline{\mathcal{M}}_{2,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{LS}_{12} \quad \text{and} \quad H^{15}(\overline{\mathcal{M}}_{2,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{L}^2\mathsf{S}_{12}.$$

Proof. For degree 13, consider the right exact sequence

$$H^{11}(\widetilde{\partial\mathcal{M}_{2,n}}) \to H^{13}(\overline{\mathcal{M}}_{2,n}) \to W_{13}H^{13}(\mathcal{M}_{2,n}) \to 0.$$

By [8], the semisimplification of the left-hand side is a sum of terms S_{12}. Proposition 2.7 shows that the semisimplification of the right-hand side consists only of Tate twists of S_{12}. Thus, the middle term does too. The argument for degree 15 is similar, using the degree 11 and 13 results, and the fact that $H^2(\overline{\mathcal{M}}_{g,n})$ is pure Hodge–Tate. $\square$

3. Finite generation

Let S^* be an STE. By Lemma 2.1, $S^k(\overline{\mathcal{M}}_{g,n})$ contains $H^k(\overline{\mathcal{M}}_{g,n})$ for all g and n if and only if it surjects onto $W_kH^k(\mathcal{M}_{g,n})/\big(\Phi^k_{g,n} + RH^k(\mathcal{M}_{g,n})\big)$ for all g and n. The following lemma provides a sufficient criterion for the vanishing of $W_kH^k(\mathcal{M}_{g,n})/\Phi^k_{g,n}$. The argument is similar to the proof of Proposition 2.2, using the Künneth decomposition of the Leray spectral sequence for the n-fold fiber product $\mathcal{C}^n \to \mathcal{M}_g$.

Lemma 3.1. *If $n > k$, or if $n \geq k$ and $g \geq 2$, then $W_kH^k(\mathcal{M}_{g,n}) = \Phi^k_{g,n}$.*

Proof. When $g = 0$, Keel [16] proved that all cohomology is pushed forward from the boundary, so $W_kH^k(\mathcal{M}_{0,n}) = 0$ and there is nothing to prove. When $g = 1$, this is a consequence of the results of Petersen [21] recalled in Proposition 2.2 above.

Suppose $g \geq 2$, and let $\pi\colon \mathcal{C} \to \mathcal{M}_g$ be the universal curve. We must show that $\Phi^k_{g,n} = W_kH^k(\mathcal{M}_{g,n})$, for $n \geq k$. There is an open inclusion $\mathcal{M}_{g,n} \subset \mathcal{C}^n$, and hence restriction gives a surjection from $W_kH^k(\mathcal{C}^n)$ to $W_kH^k(\mathcal{M}_{g,n})$. Given a subset $A \subset \{1, \ldots, n\}$, let $\mathcal{C}^n \to \mathcal{C}^A$ be the projection onto the factors indexed by A. It will suffice to show that $H^k(\mathcal{C}^n)$ is generated by pullbacks from $H^k(\mathcal{C}^A)$, as A runs over sets of size less than k. Since $f\colon \mathcal{C}^n \to \mathcal{M}_g$ is smooth and proper, the Leray spectral sequence degenerates at E_2, and

$$H^k(\mathcal{C}^n) = \bigoplus_{p+q=k} H^p(\mathcal{M}_g, R^qf_*\mathbb{Q}).$$

Applying the Künneth formula, we obtain

$$H^k(\mathcal{C}^n) = \bigoplus_{\substack{p+q=k \\ i_1+\ldots+i_n=q}} H^p\left(\mathcal{M}_g, R^{i_1}\pi_*\mathbb{Q} \otimes \cdots \otimes R^{i_n}\pi_*\mathbb{Q}\right). \tag{3.1}$$

Then each summand corresponding to $(i_1, \ldots, i_n)$ on the right hand side is pulled back from $\mathcal{C}^A$, for $A = \{j \mid i_j > 0\}$, which has size at most k. This proves the lemma for $n > k$.

For $n = k$, note that the only summand on the right hand side for which i_j is positive for all j is $H^0(\mathcal{M}_g, R^1\pi_*\mathbb{Q} \otimes \cdots \otimes R^1\pi_*\mathbb{Q})$. This is identified with the monodromy invariant subspace of the tensor product of k copies of $H^1(C)$, where C is a fiber of π. We claim that this monodromy invariant subspace vanishes. To see this, choose simple closed curves giving a symplectic basis $\gamma_1, \ldots, \gamma_{2g}$ for $H_1(C)$, with $\langle\gamma_i, \gamma_{g+i}\rangle = 1$. Consider the action of Dehn twists around these curves on the induced basis for $H^1(C) \otimes \cdots \otimes H^1(C)$. If T_j is the action of Dehn twist around γ_j, then $T_j(\gamma_\ell) = \gamma_\ell$, unless $\ell = g+j$, in which case $T(\gamma_{g+j}) = \gamma_{g+j} + \gamma_j$, where subscripts are taken modulo $2g$. If

$$x = \sum a_{\ell_1,\ldots,\ell_{2g}} \gamma_{\ell_1} \otimes \cdots \otimes \gamma_{\ell_{2g}} = \sum a_{\ell_1,\ldots,\ell_{2g}} T_j(\gamma_{\ell_1}) \otimes \cdots \otimes T_j(\gamma_{\ell_{2g}})$$

is T_j-invariant, then $a_{\ell_1,\ldots,\ell_{2g}} = 0$ if $\ell_m = j + g$ for any m. Varying j, we see that if x is invariant under the action of all Dehn twists, then $x = 0$. □

Proof of Theorem 1.3. Let S^* be the STE generated by the cohomology groups listed in the statement of the theorem. By Lemma 2.1, it suffices to check that $S^*(\overline{\mathcal{M}}_{g',n'})$ surjects onto $W_{k'}H^{k'}(\overline{\mathcal{M}}_{g',n'})/\left(\Phi^k_{g',n'} + RH^{k'}(\mathcal{M}_{g',n'})\right)$ for all (g', n', k') with $k' \leq k$. By Lemma 3.1, the target vanishes when $n' > k'$. By the vcd of $\overline{\mathcal{M}}_{g',n'}$, it also vanishes when $k' > 4g' - 4 + n'$. Finally, $H^{k'}(\mathcal{M}_{g',n'})$ is tautological for $k' \leq \frac{2g'-2}{3}$ [32]. □

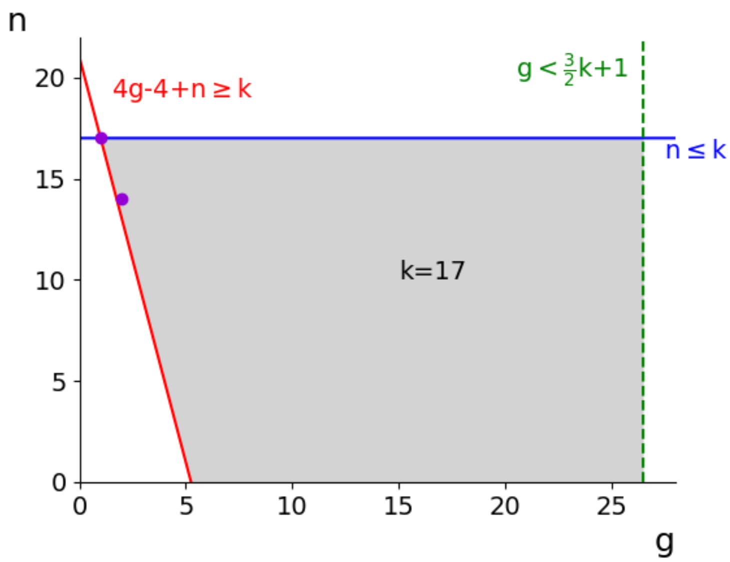


FIGURE 1. The argument in the proof of Theorem 1.3 shows that $W_{17}H^{17}(\mathcal{M}_{g,n})/(\Phi^{17}_{g,n}+RH^{17}(\mathcal{M}_{g,n}))$ vanishes for (g,n) outside the gray shaded region. Note that this quotient does *not* vanish for (g,n) equal to $(1,17)$ and $(2,14)$, which are pictured by purple dots.

4. The Chow–Künneth generation Property

In this section we prove a key lemma about the cycle class map for spaces that have the following property.

Definition 4.1. Let X be a smooth algebraic stack of finite type over a field, stratified by quotient stacks. We say that X has the *Chow–Künneth generation Property* (CKgP) if for all algebraic stacks Y (of finite type, stratified by quotient stacks) the exterior product

$$A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$$

is surjective.

For convenience, we record here several properties of the CKgP, all of which are proven in [6, Section 3.1].

Proposition 4.2. *Let X be a smooth algebraic stack of finite type over a field, stratified by quotient stacks.*

1. *if $U \subset X$ is open and X has the CKgP, then U has the CKgP;*
2. *if $Y \to X$ is proper, surjective, representable by DM stacks, and Y has the CKgP, then X has the CKgP;*
3. *if X admits a finite stratification $X = \coprod_{S \in \mathcal{S}} S$ such that each S has the CKgP, then X has the CKgP;*
4. *if $V \to X$ is an affine bundle, then V has the CKgP if and only if X has the CKgP;*
5. *if X has the CKgP, and $G \to X$ is a Grassmann bundle, then G has the CKgP;*
6. *if $X = \mathrm{BGL}_n, \mathrm{BSL}_n,$ or BPGL_n, then X has the CKgP.*

If X is smooth and proper and has the CKgP, then the cycle class map for X is an isomorphism [6, Lemma 3.11].

When X is smooth, but not necessarily proper (e.g., for $X = \mathcal{M}_{g,n}$), the cycle class map is not necessarily an isomorphism. Nevertheless, we have the following useful substitute (cf. [18, 30] for slightly different statements with similar proofs).

Lemma 4.3. *Let X be an open substack of a smooth proper Deligne–Mumford stack $\overline{X}$ over the complex numbers. If X has the CKgP, then the cycle class map*

$$\mathrm{cl}\colon \bigoplus_i A_i(X) \to \bigoplus_k W_k H^k(X)$$

is surjective. In particular, if k is odd then $W_k H^k(X) = 0$, and if k is even then $W_k H^k(X)$ is pure Hodge–Tate.

Proof. Set $d := \dim \overline{X}$ and $D := \overline{X} \smallsetminus X$. Let $\Delta \subset \overline{X} \times \overline{X}$ denote the diagonal. Because X has the CKgP, the exterior product map

$$\bigoplus_{\ell=0}^{d} A^\ell(\overline{X}) \otimes A^{d-\ell}(X) \to A^d(\overline{X} \times X)$$

is surjective. We have the excision exact sequence

$$A^d(\overline{X} \times D) \to A^d(\overline{X} \times \overline{X}) \to A^d(\overline{X} \times X) \to 0.$$

It follows that we can write the class of the diagonal in $A^d(\overline{X} \times \overline{X})$ as

$$\Delta = \Gamma + \Delta^0 + \Delta^1 + \cdots + \Delta^d, \tag{4.1}$$

where Γ is supported on $\overline{X} \times D$ and each Δ^ℓ is a linear combination of cycles of the form $V^\ell \times W^{d-\ell}$, where V^ℓ and $W^{d-\ell}$ are subvarieties of $\overline{X}$ of codimension ℓ and $d-\ell$, respectively.

Let p_1 and p_2 be the projections of $\overline{X} \times \overline{X}$ to the first and second factors respectively. Given a class $\Psi \in H^*(\overline{X} \times \overline{X})$, we write $\Psi_*\colon H^*(\overline{X}) \to H^*(\overline{X})$ for the associated correspondence, defined by $\Psi_* \alpha = p_{2*}(p_1^* \alpha \cdot \Psi)$.

Let $a \in W_k H^k(X)$. Let α be a lift of a in $W_k H^k(\overline{X}) = H^k(\overline{X})$. Then

$$\alpha = \Delta_* \alpha = (\Gamma_* + \Delta^0_* + \Delta^1_* + \cdots + \Delta^d_*)\alpha.$$

First, we study $\Delta^\ell_* \alpha = p_{2*}(p_1^* \alpha \cdot \Delta^\ell)$. Note that $p_1^* \alpha \cdot \Delta^\ell$ vanishes for dimension reasons if $k + 2\ell > 2d$. Furthermore, since p_2 is of relative dimension d, the pushforward by p_2 of any cycle will vanish if $k + 2\ell < 2d$. Thus, the only non-zero terms occur when $k = 2d - 2\ell$. Moreover, $\Delta^\ell_* \alpha$ is a linear combination of the form $\sum a_i W^{d-\ell}$, so it lies in the image of the cycle class map.

Next, we study $\Gamma_* \alpha$. It suffices to treat the case that Γ is the class of a subvariety of $\overline{X} \times \overline{X}$ contained in $\overline{X} \times D$, as it is a linear combination of such subvarieties. In this case, the map $p_2|_\Gamma\colon \Gamma \to \overline{X}$ factors through $D \to \overline{X}$. Thus, $\Gamma_* \alpha$ maps to zero under the restriction to $W_k H^k(X)$ because the correspondence map factors through the cohomology of the boundary D. Thus, $\alpha|_X = a$ is in the image of the cycle class map. $\square$

Remark 4.4. Essentially the same argument (only with $\mathbb{Q}_\ell$-coefficients) gives a similar statement for the cycle class map to ℓ-adic étale cohomology over an arbitrary field.

The first two authors have previously given many examples of of moduli spaces $\mathcal{M}_{g,n}$ that have the CKgP and also satisfy $A^*(\mathcal{M}_{g,n}) = R^*(\mathcal{M}_{g,n})$ [6, Theorem 1.4]. For the inductive arguments in this paper, we need more base cases in genus 7. In the next section, we prove that $\mathcal{M}_{7,n}$ has the CKgP and $A^*(\mathcal{M}_{7,n}) = R^*(\mathcal{M}_{7,n})$, for $n \leq 3$. The table below records the previously known results from [6, Theorem 1.4] together with Theorem 1.10.

g	0	1	2	3	4	5	6	7
$c(g)$	∞	10	10	11	11	7	5	3

TABLE 1. $\mathcal{M}_{g,n}$ has the CKgP and $A^*(\mathcal{M}_{g,n}) = R^*(\mathcal{M}_{g,n})$, for $n \leq c(g)$, by [6, Theorem 1.4] and Theorem 1.10.

The following is a consequence of Lemma 4.3.

Proposition 4.5. *For all $g \leq 7$ and $n \leq c(g)$ as specified in Table 1, we have*

$$W_k H^k(\mathcal{M}_{g,n}) = RH^k(\mathcal{M}_{g,n}).$$

In particular, if k is odd and $n \leq c(g)$, then

$$W_k H^k(\mathcal{M}_{g,n}) = \operatorname{gr}_k^W H_c^k(\mathcal{M}_{g,n}) = 0.$$

Proof of Theorem 1.9, assuming Theorem 1.1. By the Hard Lefschetz theorem, it suffices to prove that $H^k(\overline{\mathcal{M}}_{3,n})^{\mathrm{ss}}$ is a polynomial in L and S_{12} for fixed $k \leq 17$ and $n \leq 11$. Consider the right exact sequence

$$H^{k-2}(\widetilde{\partial \mathcal{M}_{3,n}}) \to H^k(\overline{\mathcal{M}}_{3,n}) \to W_k H^k(\mathcal{M}_{3,n}) \to 0.$$

By Theorem 1.1 and [8, Theorems 1.1, 1.2 and Lemma 2.1], for $k' \leq 15$, $H^{k'}(\overline{\mathcal{M}}_{g',n'})^{\mathrm{ss}}$ is a polynomial in L and S_{12} for all $g' \neq 1$ and all n' as well as for $g' = 1$ and $n' \leq 14$. It follows that, $H^{k-2}(\widetilde{\partial \mathcal{M}_{3,n}})^{\mathrm{ss}}$ is a polynomial in L and S_{12}. By [6, Theorem 1.4], $\mathcal{M}_{3,n}$ has the CKgP and $A^*(\mathcal{M}_{3,n}) = R^*(\mathcal{M}_{3,n})$ for $n \leq 11$. Thus $W_k H^k(\mathcal{M}_{3,n}) = RH^k(\mathcal{M}_{3,n})$ by Lemma 4.3. Hence, $W_k H^k(\mathcal{M}_{3,n})$ is pure Hodge–Tate, and the theorem follows. $\square$

5. THE CKGP IN GENUS 7 WITH AT MOST 3 MARKED POINTS

In order to prove Theorems 1.1, 1.5(3) for $k = 14$, and 1.7, we need more base cases in genus 7. The required base cases are given by Theorem 1.10, which we now prove.

In order to prove Theorem 1.10, we filter $\mathcal{M}_{7,n}$ by gonality:

$$\mathcal{M}^2_{7,n} \subset \mathcal{M}^3_{7,n} \subset \mathcal{M}^4_{7,n} \subset \mathcal{M}^5_{7,n}.$$

Here, $\mathcal{M}^k_{7,n}$ is the locus parametrizing smooth curves C with n marked points that admit a map of degree at most k to $\mathbb{P}^1$. Standard results from Brill–Noether theory show that the maximal gonality of a genus 7 curve is 5. By Proposition 4.2(3), to show that $\mathcal{M}_{7,n}$ has the CKgP, it suffices to show that each gonality stratum

$$\mathcal{M}^k_{7,n} \smallsetminus \mathcal{M}^{k-1}_{7,n}.$$

has the CKgP. Moreover, to show that $A^*(\mathcal{M}_{7,n}) = R^*(\mathcal{M}_{7,n})$, it suffices to show for each k that all classes supported on $\mathcal{M}^k_{7,n}$ are tautological up to classes supported on $\mathcal{M}^{k-1}_{7,n}$. In other words, we must show that every class in $A^*(\mathcal{M}^k_{7,n} \smallsetminus \mathcal{M}^{k-1}_{7,n})$ pushes forward to a class in $A^*(\mathcal{M}_{7,n} \smallsetminus \mathcal{M}^{k-1}_{7,n})$ that is the restriction of a tautological class on $\mathcal{M}_{7,n}$.

5.1. **Hyperelliptic and trigonal loci.** By [6, Lemma 9.9], if $n \leq 14$, then $\mathcal{M}^3_{7,n}$ has the CKgP and all classes in $A^*(\mathcal{M}_{7,n})$ supported on $\mathcal{M}^3_{7,n}$ are tautological. Note that this includes the hyperelliptic locus.

5.2. **The tetragonal locus.** To study the tetragonal locus $\mathcal{M}^4_{7,n} \smallsetminus \mathcal{M}^3_{7,n}$, we will use the Hurwitz stack $\mathcal{H}_{4,g,n}$ parametrizing degree 4 covers $f\colon C \to \mathbb{P}^1$, where C is a smooth curve of genus g with n marked points. There is a forgetful morphism

$$\beta_n\colon \mathcal{H}_{4,g,n} \to \mathcal{M}_{g,n}.$$

Restricting to curves of gonality exactly 4, we obtain a proper morphism

$$\beta_n'\colon \mathcal{H}^{\lozenge}_{4,g,n} := \mathcal{H}_{4,g,n} \smallsetminus \beta_n^{-1}(\mathcal{M}^3_{g,n}) \to \mathcal{M}_{g,n} \smallsetminus \mathcal{M}^3_{g,n}$$

with image $\mathcal{M}^4_{g,n} \smallsetminus \mathcal{M}^3_{g,n}$. To show that $\mathcal{M}^4_{7,n} \smallsetminus \mathcal{M}^3_{7,n}$ has the CKgP, it suffices to show that $\mathcal{H}^{\lozenge}_{4,7,n}$ has the CKgP by Proposition 4.2(2). We will do so by further stratifying $\mathcal{H}^{\lozenge}_{4,7,n}$.

In [5, Section 4.4], the first two authors studied a stratification of $\mathcal{H}_{4,7} := \mathcal{H}_{4,7,0}$ with no markings. Here we carry out a similar analysis with marked points. The Casnati–Ekedahl structure theorem [9] associates to a point in $\mathcal{H}_{4,g}$ a rank 3 vector bundle E and a rank 2 vector bundle F on $\mathbb{P}^1$, both of degree $g+3$, equipped with a canonical isomorphism $\det E \cong \det F$ [7, Section 3]. Let $\mathcal{B}$ be the moduli stack of pairs of vector bundles (E, F) on $\mathbb{P}^1$ of degree $g+3$, together with an isomorphism of their determinants as in [7, Definition 5.2]. Let $\pi\colon \mathcal{P} \to \mathcal{B}$ be the universal $\mathbb{P}^1$-fibration, and let $\mathcal{E}$ and $\mathcal{F}$ be the universal bundles on $\mathcal{P}$. There is a natural morphism $\mathcal{H}_{4,g} \to \mathcal{B}$ that sends a degree 4 cover to its associated pair of vector bundles. Moreover, the Casnati–Ekedahl construction gives an embedding of the universal curve $\mathcal{C}$ over $\mathcal{H}_{4,g}$ into $\mathbb{P}\mathcal{E}^\vee$.

Consider the natural commutative diagram

$$\begin{tikzcd} \mathcal{C}_n \arrow[r, "b"] \arrow[d] & \mathcal{C} \arrow[r, "a"] \arrow[d] & \mathbb{P}\mathcal{E}^\vee \arrow[d] \\ \mathcal{H}_{4,g,n} \arrow[r] \arrow[u, bend left, "\sigma_i"] & \mathcal{H}_{4,g} \arrow[r] & \mathcal{B}, \end{tikzcd} \tag{5.1}$$

where $\mathcal{C}_n$ is the universal curve over $\mathcal{H}_{4,g,n}$. For each i, the map $a \circ b \circ \sigma_i$ sends a pointed curve to the image of the ith marking under the Casnati–Ekedahl embedding. Taking the product of these maps for $i = 1, \ldots, n$, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{H}^{\lozenge}_{4,g,n} \arrow[r] \arrow[d] & \mathcal{H}_{4,g,n} \arrow[r] \arrow[d] & (\mathbb{P}\mathcal{E}^\vee)^n \arrow[d] \\ \mathcal{H}^{\lozenge}_{4,g} \arrow[r] & \mathcal{H}_{4,g} \arrow[r] & \mathcal{B}. \end{tikzcd} \tag{5.2}$$

Lemma 5.1 (Lemma 10.5 of [6])**.** *Suppose $x \in A^*(\mathcal{H}^{\lozenge}_{4,g,n})$ lies in the image of the map $A^*((\mathbb{P}\mathcal{E}^\vee)^n) \to A^*(\mathcal{H}^{\lozenge}_{4,g,n})$. Then $\beta'_{n*}x$ is tautological on $\mathcal{M}_{g,n} \smallsetminus \mathcal{M}^3_{g,n}$.*

The splitting types of the Casnati–Ekedahl bundles E and F induce a stratification on $\mathcal{H}_{4,7}$. We write $E = (e_1, e_2, e_3)$ and $F = (f_1, f_2)$ to indicate that the bundles have splitting types

$$E = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2) \oplus \mathcal{O}(e_3) \qquad \text{and} \qquad F = \mathcal{O}(f_1) \oplus \mathcal{O}(f_2).$$

We will consider a stratification into three pieces

$$\mathcal{H}_{4,7,n} = X_n \sqcup Y_n \sqcup Z_n.$$

The three strata correspond to unions of splitting types of E and F. The possible splitting types are recorded in [5, Section 4.4]. The locus Z_n is the set of covers with maximally unbalanced splitting types and parametrizes hyperelliptic curves [5, Equation 4.5]. Its image in $\mathcal{M}_{7,n}$ is contained in $\mathcal{M}^3_{7,n}$, which has the CKgP when $n \leq 14$, as noted above. We will show that X_n and Y_n have the CKgP and that $A^*((\mathbb{P}\mathcal{E}^\vee)^n)$ surjects onto the Chow ring of their union, which is $\mathcal{H}^\lozenge_{4,7,n}$. We start with X_n.

Let $X_n \subset \mathcal{H}_{4,7,n}$ denote the locus of covers with splitting types $E = (3,3,4)$ and $F = (5,5)$, or $E = (3,3,4)$ and $F = (4,6)$.

Lemma 5.2. *If $n \leq 3$, then X_n has the CKgP and $A^*((\mathbb{P}\mathcal{E}^\vee)^n) \to A^*(X_n)$ is surjective.*

Proof. Taking $f = 4$ in [6, Definition 10.8], we have $X_n = \mathcal{H}^4_{4,7,n}$. The result then follows from [6, Lemmas 10.11 and 10.12] with $g = 7$ and $f = f_1 = 4$. ☐

Now let $Y_n \subset \mathcal{H}_{4,7,n}$ be the union

$$Y_n = \Sigma_{2,n} \sqcup \Sigma_{3,n},$$

where $\Sigma_{2,n}$ parametrizes covers with splitting types $E = (2,4,4)$ and $F = (4,6)$, and $\Sigma_{3,n}$ parametrizes covers with splitting types $E = (2,3,5)$ and $F = (4,6)$.

Recall that $\pi\colon \mathcal{P} \to \mathcal{B}$ is the structure map for the $\mathbb{P}^1$ bundle $\mathcal{P}$. Let $\gamma\colon \mathbb{P}\mathcal{E}^\vee \to \mathcal{P}$ denote the structure map and $\eta_i\colon (\mathbb{P}\mathcal{E}^\vee)^n \to \mathbb{P}\mathcal{E}^\vee$ denote the ith projection. Define

$$z_i := \eta_i^*\gamma^*c_1(\mathcal{O}_{\mathcal{P}}(1)) \quad \text{and} \quad \zeta_i := \eta_i^* c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1)).$$

Then z_i and ζ_i generate $A^*((\mathbb{P}\mathcal{E}^\vee)^n)$ as an algebra over $A^*(\mathcal{B})$. Write $\Sigma_\ell := \Sigma_{\ell,0}$.

Lemma 5.3. *For $\ell = 2,3$ and $n \leq 3$, there is a surjection*

$$A^*(\Sigma_\ell)[z_1,\dots,z_n,\zeta_1,\dots,\zeta_n] \to A^*(\Sigma_{\ell,n})$$

induced by $\Sigma_{\ell,n} \to \Sigma_\ell$ and restriction from $(\mathbb{P}\mathcal{E}^\vee)^n$. Moreover, $\Sigma_{\ell,n}$ has the CKgP.

Proof. Let $\vec{e}$ and $\vec{f}$ be the splitting types associated to Σ_ℓ and let $\mathcal{B}_{\vec{e},\vec{f}} \subset \mathcal{B}$ be the locally closed substack that parametrizes pairs of bundles (E,F) with locally constant splitting types $\vec{e}$ and $\vec{f}$. We write $\mathcal{O}(\vec{e}) := \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_k)$. By construction, $\Sigma_{\ell,n}$ is the preimage of $\mathcal{B}_{\vec{e},\vec{f}}$ along $\mathcal{H}_{4,7,n} \to \mathcal{B}$. We therefore study the base change of (5.2) along $\mathcal{B}_{\vec{e},\vec{f}} \to \mathcal{B}$:

$$\begin{tikzcd} \Sigma_{\ell,n} \arrow[r] \arrow[d] & (\mathbb{P}\mathcal{E}^\vee|_{\mathcal{B}_{\vec{e},\vec{f}}})^n \arrow[d] \\ \Sigma_\ell \arrow[r] & \mathcal{B}_{\vec{e},\vec{f}}. \end{tikzcd} \tag{5.3}$$

We now recall the description of Σ_ℓ as an open substack of a vector bundle on $\mathcal{B}_{\vec{e},\vec{f}}$, as in [5, Lemma 3.10]. Let

$$U \subset H^0(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})) = H^0(\mathbb{P}\mathcal{O}(\vec{e})^\vee, \gamma^*\mathcal{O}(\vec{f})^\vee \otimes \mathcal{O}_{\mathbb{P}\mathcal{O}(\vec{e})^\vee}(2))$$

be the open subset of equations that define a smooth curve, as in [5, Lemma 3.10]. Then

$$\Sigma_\ell = [(U \times \mathbb{G}_m)/\operatorname{SL}_2 \ltimes (\operatorname{Aut}(\mathcal{O}(\vec{e})) \times \operatorname{Aut}(\mathcal{O}(\vec{f})))].$$

The stack $\mathcal{B}_{\vec{e},\vec{f}}$ is the part obtained by forgetting U:

$$\mathcal{B}_{\vec{e},\vec{f}} = [\mathbb{G}_m/\operatorname{SL}_2 \ltimes (\operatorname{Aut}(\mathcal{O}(\vec{e})) \times \operatorname{Aut}(\mathcal{O}(\vec{f})))].$$

As explained in [5, Equation 3.1], there is a product of stacks BGL_n which is an affine bundle over $\mathrm{BAut}(\mathcal{O}(\vec{e}))$. As such, $\mathcal{B}_{\vec{e},\vec{f}}$ has CKgP by Proposition 4.2(4) and (6). It follows that Σ_ℓ also has the CKgP by Proposition 4.2(1) and (4).

Let us define the rank 2 vector bundle $\mathcal{W} := \gamma^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(2)$ on $\mathbb{P}\mathcal{E}^\vee$. Write $\mathcal{W}_{\vec{e},\vec{f}}$ for the restriction of $\mathcal{W}$ to $\mathbb{P}\mathcal{E}^\vee|_{\mathcal{B}_{\vec{e},\vec{f}}}$. The discussion above says that Σ_ℓ is an open substack of the vector bundle $(\pi \circ \gamma)_*\mathcal{W}_{\vec{e},\vec{f}}$ on $\mathcal{B}_{\vec{e},\vec{f}}$.

Next, we give a similar description with marked points. Consider the evaluation map

$$(\pi \circ \gamma)^*(\pi \circ \gamma)_*\mathcal{W}_{\vec{e},\vec{f}} \to \mathcal{W}_{\vec{e},\vec{f}}.$$

Pulling back to the fiber product $(\mathbb{P}\mathcal{E}^\vee|_{\mathcal{B}_{\vec{e},\vec{f}}})^n$, we obtain

$$\eta_j^*(\pi \circ \gamma)^*(\pi \circ \gamma)_*\mathcal{W}_{\vec{e},\vec{f}} \to \bigoplus_{i=1}^n \eta_i^*\mathcal{W}_{\vec{e},\vec{f}}. \tag{5.4}$$

Note that $\eta_j^*(\pi \circ \gamma)^*(\pi \circ \gamma)_*\mathcal{W}_{\vec{e},\vec{f}}$ is independent of j. The kernel $\mathcal{Y}$ of (5.4) parametrizes tuples of n points on $\mathbb{P}\mathcal{O}(\vec{e})^\vee$ together with a section of

$$H^0(\mathbb{P}\mathcal{O}(\vec{e})^\vee, \gamma^*\mathcal{O}(\vec{f})^\vee \otimes \mathcal{O}_{\mathbb{P}\mathcal{O}(\vec{e})^\vee}(2))$$

that vanishes on the points. There is therefore a natural map $\Sigma_{\ell,n} \hookrightarrow \mathcal{Y}$ defined by sending a pointed curve to the images of the points on $\mathbb{P}\mathcal{O}(\vec{e})^\vee$ and the equation of the curve. The kernel $\mathcal{Y}$ is not locally free. Nevertheless, its restriction to the open substack $U_n \subset (\mathbb{P}\mathcal{E}^\vee|_{\mathcal{B}_{\vec{e},\vec{f}}})^n$ where (5.4) is surjective is locally free.

We claim that the map $\Sigma_{\ell,n} \to \mathcal{Y}$ factors through $\mathcal{Y}|_{U_n}$. To see this, suppose $C \subset \mathbb{P}\mathcal{O}(\vec{e})^\vee$ is the vanishing of a section of $\gamma^*\mathcal{O}(\vec{f}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{O}(\vec{e})^\vee}(2)$ and C is smooth and irreducible. By [6, Lemma 10.6], if $n \leq 3$, the evaluation map (5.4) is surjective at any tuple of n distinct points on C. It follows that the image of the composition $\Sigma_{\ell,n} \to \mathcal{Y} \to (\mathbb{P}\mathcal{E}^\vee|_{\mathcal{B}_{\vec{e},\vec{f}}})^n$ is contained in U_n. Hence, $\Sigma_{\ell,n} \hookrightarrow \mathcal{Y}|_{U_n}$.

In summary, we have a sequence of maps

$$\Sigma_{\ell,n} \hookrightarrow \mathcal{Y}|_{U_n} \to U_n \hookrightarrow (\mathbb{P}\mathcal{E}^\vee|_{\mathcal{B}_{\vec{e},\vec{f}}})^n \to (\mathcal{P}|_{\mathcal{B}_{\vec{e},\vec{f}}})^n \to \mathcal{B}_{\vec{e},\vec{f}}.$$

each of which is an open inclusion, vector bundle, or product of projective bundles. Since $\mathcal{B}_{\vec{e},\vec{f}}$ has the CKgP, it follows that $\Sigma_{\ell,n}$ also has the CKgP by Proposition 4.2(1),(4), and (5). Moreover, we see that

$$A^*(\mathcal{B}_{\vec{e},\vec{f}})[z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n] \to A^*(\Sigma_{\ell,n})$$

is surjective. Finally note that $\Sigma_{\ell,n} \to \mathcal{B}_{\vec{e},\vec{f}}$ factors through Σ_ℓ, so $A^*(\mathcal{B}_{\vec{e},\vec{f}}) \to A^*(\Sigma_{\ell,n})$ factors through $A^*(\Sigma_\ell)$. This proves the claim. $\square$

Corollary 5.4. *For $n \leq 3$, $\mathcal{H}^\diamondsuit_{4,7,n}$ has the CKgP. Hence, $\mathcal{M}^4_{7,n}$ has the CKgP.*

Proof. We have $\mathcal{H}^\diamondsuit_{4,7,n} = X_n \sqcup Y_n$. By Lemmas 5.2 and 5.3, each of these pieces has the CKgP. Note that $\mathcal{H}^\diamondsuit_{4,7,n}$ maps properly onto $\mathcal{M}^4_{7,n} \smallsetminus \mathcal{M}^3_{7,n}$ and $\mathcal{M}^3_{7,n}$ has the CKgP. Thus the result follows by Proposition 4.2(2)-(3). $\square$

The next step is to show that all classes in $A^*(\Sigma_{\ell,n})$ are restrictions from $(\mathbb{P}\mathcal{E}^\vee)^n$.

Lemma 5.5. *For $\ell = 2, 3$ and $n \leq 3$, the restriction $A^*((\mathbb{P}\mathcal{E}^\vee)^n) \to A^*(\Sigma_{\ell,n})$ is surjective.*

Proof. Consider the following diagram:

$$\begin{tikzcd} A^*((\mathbb{P}\mathcal{E}^\vee)^n) \arrow[r] & A^*(\Sigma_{\ell,n}) \\ A^*(\mathcal{B})[z_1, \dots, z_n, \zeta_1, \dots, \zeta_n] \arrow[u] \arrow[r] & A^*(\Sigma_\ell)[z_1, \dots, z_n, \zeta_1, \dots, \zeta_n]. \arrow[u] \end{tikzcd} \tag{5.5}$$

Note that the map $A^*(\mathcal{B}) \to A^*(\Sigma_\ell)$ is surjective by [5, Lemma 4.2] for $\ell = 2$, and by [5, Lemma 4.3(1)] for $\ell = 3$. Therefore, the bottom horizontal arrow is surjective. By Lemma 5.3, the right vertical arrow is also surjective. Hence, the top horizontal arrow is surjective. □

Lemma 5.6. *For $n \leq 3$, the pullback map $A^*((\mathbb{P}\mathcal{E}^\vee)^n) \to A^*(\mathcal{H}^\lozenge_{4,7,n})$ is surjective.*

Proof. First, we fix some notation. Let

$$\Sigma_{3,n} \overset{\jmath}{\hookrightarrow} Y_n \overset{\iota}{\hookrightarrow} \mathcal{H}^\lozenge_{4,7,n}$$

denote the natural closed inclusion maps. Let $\phi \colon \mathcal{H}^\lozenge_{4,7,n} \to (\mathbb{P}\mathcal{E}^\vee)^n$ be the map from the top left to top right in diagram (5.1). We let $\phi' : X_n \to (\mathbb{P}\mathcal{E}^\vee)^n$ be composite of the open inclusion $X_n \hookrightarrow \mathcal{H}^\lozenge_{4,7,n}$ and ϕ. Let $\psi := \phi \circ \iota$, and let ψ' be the composite of the open inclusion $\Sigma_{2,n} \hookrightarrow Y_n$ and ψ.

Consider the following commutative diagram, where the bottom row is exact.

$$\begin{tikzcd} & A^*((\mathbb{P}\mathcal{E}^\vee)^n) \arrow[d, "\phi^*"'] \arrow[dr, "\phi'^*"] & & \\ A^{*-2}(Y_n) \arrow[r, "\iota_*"] & A^*(\mathcal{H}^\lozenge_{4,7,n}) \arrow[r] & A^*(X_n) \arrow[r] & 0. \end{tikzcd} \tag{5.6}$$

By Lemma 5.2, ϕ'^* is surjective. It thus suffices to show that the image of ι_* is contained in the image of ϕ^*. To do so, we consider another commutative diagram where the bottom row is exact.

$$\begin{tikzcd} & A^*((\mathbb{P}\mathcal{E}^\vee)^n) \arrow[d, "\psi^*"'] \arrow[dr, "\psi'^*"] & & \\ A^{*-1}(\Sigma_{3,n}) \arrow[r, "\jmath_*"] & A^*(Y_n) \arrow[r] & A^*(\Sigma_{2,n}) \arrow[r] & 0. \end{tikzcd} \tag{5.7}$$

By Lemma 5.5, ψ'^* is surjective. Moreover, by the projection formula and Lemma 5.5, the image of $\jmath$ is generated as an $A^*((\mathbb{P}\mathcal{E}^\vee)^n)$ module by the fundamental class $[\Sigma_{3,n}] \in A^*(Y_n)$. Therefore, any class $\alpha \in A^*(Y_n)$ can be written as

$$\alpha = \psi^*\alpha_0 + [\Sigma_{3,n}]\psi^*\alpha_1 = \iota^*\phi^*\alpha_0 + [\Sigma_{3,n}]\iota^*\phi^*\alpha_1,$$

where $\alpha_i \in A^*((\mathbb{P}\mathcal{E}^\vee)^n)$. By the projection formula,

$$\iota_*\alpha = [Y_n]\phi^*\alpha_0 + [\Sigma_{3,n}]\phi^*\alpha_1,$$

where now the fundamental class $[\Sigma_{3,n}]$ is a class on $\mathcal{H}^\lozenge_{4,7,n}$. It thus suffices to show that the classes $[Y_n]$ and $[\Sigma_{3,n}]$ are in the image of ϕ^*.

By [5, Lemma 4.8], $[\overline{\Sigma_\ell}]$ is in the image of $A^*(\mathcal{B}) \to A^*(\mathcal{H}^{\Diamond}_{4,7})$. Because $[\Sigma_{3,n}]$ is the pullback of $[\Sigma_3]$ along $A^*(\mathcal{H}^{\Diamond}_{4,7}) \to A^*(\mathcal{H}^{\Diamond}_{4,7,n})$ and $[Y_n]$ is the pullback of $[\overline{\Sigma_2}]$, both $[\Sigma_{3,n}]$ and $[Y_n]$ are in the image of $A^*(\mathcal{B}) \to A^*(\mathcal{H}^{\Diamond}_{4,7,n})$. Hence, they are in the image of ϕ^*. □

Recall that proper, surjective maps induce surjective maps on rational Chow groups. Since the map $\beta_n' \colon \mathcal{H}^{\Diamond}_{4,7,n} \to \mathcal{M}_{7,n} \smallsetminus \mathcal{M}^3_{7,n}$ is proper with image $\mathcal{M}^4_{7,n} \smallsetminus \mathcal{M}^3_{7,n}$, every class supported on the tetragonal locus is the pushforward of a class from $\mathcal{H}^{\Diamond}_{4,7,n}$. Combining Lemmas 5.1 and 5.6 therefore proves the following.

Lemma 5.7. *If $n \leq 3$, then all classes supported on $\mathcal{M}^4_{7,n} \smallsetminus \mathcal{M}^3_{7,n}$ are tautological.*

5.3. **The pentagonal locus.** It remains to study the locus $\mathcal{M}^\circ_{7,n} = \mathcal{M}_{7,n} \smallsetminus \mathcal{M}^4_{7,n}$ of curves of gonality exactly 5. Mukai showed that every curve in $\mathcal{M}^\circ_7$ is realized as a linear section of the orthogonal Grassmannian in its spinor embedding $\mathrm{OG}(5,10) \hookrightarrow \mathbb{P}^{15}$ [19]. To take advantage of this construction, we first develop a few lemmas about the orthogonal Grassmannian.

5.3.1. *The orthogonal Grassmannian.* Let $\mathcal{V}$ be the universal rank 10 bundle on BSO_{10}. The universal orthogonal Grassmannian is the quotient stack $[\mathrm{OG}(5,10)/\mathrm{SO}_{10}]$, which we think of as the orthogonal Grassmann bundle with structure map $\pi \colon \mathrm{OG}(5,\mathcal{V}) \to \mathrm{BSO}_{10}$. By construction, the pullback of $\mathcal{V}$ along π satisfies $\pi^*\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\vee$ where $\mathcal{U}$ is the universal rank 5 subbundle on $\mathrm{OG}(5,\mathcal{V})$.

Lemma 5.8. *The stack*
$$\mathrm{OG}(5,\mathcal{V}) \cong [\mathrm{OG}(5,10)/\mathrm{SO}_{10}]$$
has the CKgP. Moreover, its Chow ring is freely generated by the Chern classes of $\mathcal{U}$.

Proof. Let $V = \mathrm{span}\{e_1, \ldots, e_{10}\}$ be a fixed 10-dimensional vector space with quadratic form
$$Q = \begin{pmatrix} 0 & I_5 \\ I_5 & 0 \end{pmatrix}.$$
Let $U = \mathrm{span}\{e_1, \ldots, e_5\}$, which is an isotropic subspace. The stabilizer of SO_{10} acting on $\mathrm{OG}(5,10)$ at U is
$$\mathrm{Stab}_U = \left\{ M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : M^T Q M = Q \right\} \subset \mathrm{SO}_{10}.$$
Expanding, we have
$$M^T Q M = \begin{pmatrix} A^T & 0 \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & I_5 \\ I_5 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & A^T D \\ D^T A & B^T D + D^T B \end{pmatrix}.$$
Thus, Stab_U is defined by the conditions $D = (A^T)^{-1}$ and $B^T D + D^T B = 0$.

Note that Stab_U is a maximal parabolic subgroup and $\mathrm{OG}(5,10) = \mathrm{SO}_{10}/\mathrm{Stab}_U$. As such, the quotient $[\mathrm{OG}(5,10)/\mathrm{SO}_{10}]$ is equivalent to the classifying stack BStab_U. To gain a better understanding of the latter, consider the group homomorphism
$$\mathrm{GL}_5 \hookrightarrow \mathrm{Stab}_U, \qquad A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}.$$
For fixed D, the condition $B^T D + D^T B = 0$ is linear in B. Specifically, it says that, B lies in the $(D^T)^{-1}$ translation of the $\mathbb{A}^{10}$ of skew symmetric 5×5 matrices. In particular, the cosets of the subgroup $\mathrm{GL}_5 \hookrightarrow \mathrm{Stab}_U$ are isomorphic to affine spaces $\mathbb{A}^{10}$. In other words,

the induced map on classifying spaces $\mathrm{BGL}_5 \to \mathrm{BStab}_U \cong \mathrm{OG}(5, \mathcal{V})$ is an affine bundle. It follows that $\mathrm{OG}(5, \mathcal{V})$ has the CKgP by Proposition 4.2(4) and (6).

Furthermore, by construction, the tautological subbundle $\mathcal{U}$ on $\mathrm{OG}(5, \mathcal{V})$ pulls back to the tautological rank 5 bundle on BGL_5. It follows that $A^*(\mathrm{BStab}_U) \cong A^*(\mathrm{BGL}_5)$, and is freely generated by the Chern classes of the tautological bundle. □

Remark 5.9. There is also a natural map $\mathrm{Stab}_U \to \mathrm{GL}_5$ that sends $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ to A. The kernel of $\mathrm{Stab}_U \to \mathrm{GL}_5$ is the subgroup $G \cong (\mathbb{G}_a)^{10}$ where $A = I_5, D = I_5$ and $B + B^T = 0$. This shows that Stab_U is actually a semi-direct product $G \rtimes \mathrm{GL}_5$. The map $\mathrm{BStab}_U \to \mathrm{BGL}_5$ is a $\mathrm{B}G$-banded gerbe.

Lemma 5.10. *Let V be a rank 2ν vector bundle with quadratic form on X.*

(1) The rational Chow ring of $\mathrm{OG}(\nu, V)$ is generated over the Chow ring of X by the Chern classes of the tautological subbundle.

(2) If X has the CKgP, then $\mathrm{OG}(\nu, V)$ has the CKgP.

Proof. The argument is very similar to that for Grassmannians in type A.

First consider the case when X is a point. The orthogonal Grassmannian $\mathrm{OG}(\nu, 2\nu)$ is stratified by Schubert cells, each of which is isomorphic to an affine space. By Proposition 4.2(3), it follows that $\mathrm{OG}(\nu, 2\nu)$ has the CKgP. Moreover, the fundamental classes of these cells are expressed in terms of the Chern classes of the tautological quotient or subbundle via a Giambelli formula [17, p. 1–2]. Note that this formula involves dividing by 2, so it is important that we work with rational coefficients for this claim. In conclusion, the Chern classes of the tautological subbundle generate the Chow ring of $\mathrm{OG}(\nu, 2\nu)$.

More generally for a fiber bundle $f\colon \mathrm{OG}(\nu, V) \to X$, to prove (1), we stratify X into locally closed subsets X_i over which V is trivial, such that $\overline{X}_i \supset X_j$ for $i \geq j$ and X_i has codimension at least i. To check that the desired classes generate $A^k(\mathrm{OG}(\nu, V))$ for a given k, it suffices to show that they generate $A^k(f^{-1}(X_0 \cup X_1 \cup \cdots \cup X_k))$.

Over each piece of the stratification, $f^{-1}(X_i) = X_i \times \mathrm{OG}(\nu, 2\nu)$. Since $\mathrm{OG}(\nu, 2\nu)$ has the CKgP, the Chow ring of $f^{-1}(X_i) = X_i \times \mathrm{OG}(\nu, 2\nu)$ is generated by $A^*(X_i)$ and restrictions of the Chern classes from the tautological subbundle on $\mathrm{OG}(\nu, V)$. By excision and the push-pull formula, the Chow ring of any finite union $f^{-1}(X_0) \cup f^{-1}(X_1) \cup \cdots \cup f^{-1}(X_k)$ is generated by the desired classes.

Finally (2) follows from (1) exactly as in [6, Lemma 3.7]. □

Corollary 5.11. *For any $n \geq 1$, the n-fold fiber product*

$$\mathrm{OG}(5, \mathcal{V})^n := \mathrm{OG}(5, \mathcal{V}) \times_{\mathrm{BSO}_{10}} \cdots \times_{\mathrm{BSO}_{10}} \mathrm{OG}(5, \mathcal{V})$$

has the CKgP and its Chow ring is generated by the Chern classes of the tautological subbundles $\mathcal{U}_1, \ldots, \mathcal{U}_n$ pulled back from each factor.

Proof. The case $n = 1$ follows from Lemma 5.8. For $n > 1$, the n-fold fiber product is an orthogonal Grassmann bundle over the $(n-1)$-fold fiber product, so the claim follows from Lemma 5.10. □

Remark 5.12. By Proposition 4.2(2), the fact that $\mathrm{OG}(5, \mathcal{V})$ has the CKgP implies that BSO_{10} also has the CKgP. It should be possible to show that BSO_{10} has the CKgP (with integral coefficients as well) using the calculation of its Chow ring by Field [12]. If the calculation there holds over any field, it would show that BSO_{10} has Totaro's "weak Chow–Künneth Property," which is equivalent to the CKgP by the proof of [31, Theorem 4.1].

5.3.2. *Review of the Mukai construction.* We first review Mukai's construction and then explain how to modify it for pointed curves. The canonical model of a pentagonal genus 7 curve $C \subset \mathbb{P}^6$ lies on a 10-dimensional space of quadrics. The vector space W of these quadrics is defined by the exact sequence

$$0 \to W \to \mathrm{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2}) \to 0. \tag{5.8}$$

For each $p \in C$, the subspace $W_p \subset W$ of quadrics that are singular at p is 5-dimensional. It lies in the exact sequence (see [19, Section 3])

$$0 \to W_p \to \mathrm{Sym}^2 H^0(C, \omega_C(-p)) \to H^0(C, \omega_C(-p)^{\otimes 2}) \to 0. \tag{5.9}$$

Mukai shows that $W^\vee$ has a canonical quadratic form and $W_p^\perp$ is an isotropic subspace, so one obtains a map $C \to \mathrm{OG}(5, W^\vee)$ via $p \mapsto [W_p^\perp] \in \mathrm{OG}(5, W^\vee)$ [19, Theorem 0.4].

Let $\mathrm{OG}(5, 10) \hookrightarrow \mathbb{P}S^+$ be the spinor embedding, as in [19, Section 1]. The composition

$$C \to \mathrm{OG}(5, W^\vee) \cong \mathrm{OG}(5, 10) \hookrightarrow \mathbb{P}S^+$$

realizes C as a linear section $C = \mathrm{OG}(5, 10) \cap \mathbb{P}^6 \subset \mathbb{P}S^+$, and $C \subset \mathbb{P}^6$ is canonically embedded [19, Theorem 0.4]. Conversely, if a linear section $\mathbb{P}^6 \cap \mathrm{OG}(5, 10)$ is a smooth curve, then it is a canonically embedded pentagonal genus 7 curve [19, Proposition 2.2]. This construction works in families, and Mukai proves that there is an equivalence of stacks

$$\mathcal{M}_7^\circ \cong [\mathrm{Gr}(7, S^+) \smallsetminus \Delta / \mathrm{SO}_{10}], \tag{5.10}$$

where $\Delta \subset \mathrm{Gr}(7, S^+)$ is the closed locus of linear subspaces $\mathbb{P}^6 \subset \mathbb{P}S^+$ whose intersection with $\mathrm{OG}(5, 10)$ is not a smooth curve [19, Section 5].

The spinor representation of SO_{10} corresponds to a rank 16 vector bundle $\mathcal{S}^+$ on the classifying stack BSO_{10}. Then, the equivalence (5.10) shows that $\mathcal{M}_7^\circ$ is an open substack of the Grassmann bundle $\mathrm{Gr}(7, \mathcal{S}^+)$ over BSO_{10}:

$$\begin{tikzcd} \mathcal{M}_7^\circ \arrow[r, hook, "\alpha_0"] & \mathrm{Gr}(7, \mathcal{S}^+) \arrow[d] \\ & \mathrm{BSO}_{10}\,. \end{tikzcd} \tag{5.11}$$

We note that the tautological subbundle $\mathcal{E}$ on $\mathrm{Gr}(7, \mathcal{S}^+)$ restricts to the dual of the Hodge bundle on $\mathcal{M}_7^\circ$. In other words, if $f\colon \mathcal{C} \to \mathcal{M}_7^\circ$ is the universal curve, then $\alpha_0^*\mathcal{E} = (f_*\omega_f)^\vee$. The universal version of the Mukai construction furnishes a map $\mathcal{C} \to \mathrm{OG}(5, \mathcal{V})$ over BSO_{10}.

5.3.3. *The Mukai construction with markings.* For $n \leq 4$, we describe $\mathcal{M}_{7,n}^\circ$ in a similar fashion to (5.11), but this time as an open substack of a Grassmann bundle over $\mathrm{OG}(5, \mathcal{V})^n$. To do so, let $\iota\colon\ \mathrm{OG}(5, \mathcal{V}) \hookrightarrow \mathbb{P}\mathcal{S}^+$ be the universal spinor embedding and let $\mathcal{L} := \iota^*\mathcal{O}_{\mathbb{P}\mathcal{S}^+}(-1)$. The embedding ι is determined by an inclusion of vector bundles $\mathcal{L} \hookrightarrow \pi^*\mathcal{S}^+$ on $\mathrm{OG}(5, \mathcal{V})$. On the n-fold fiber product $\mathrm{OG}(5, \mathcal{V})^n$, let $\mathcal{L}_i$ and $\mathcal{U}_i$ denote the pullbacks of $\mathcal{L}$ and $\mathcal{U}$ respectively from the ith factor. Let $\pi_n\colon\ \mathrm{OG}(5, \mathcal{V})^n \to \mathrm{BSO}_{10}$ be the structure map. Consider the sum of the inclusions

$$\bigoplus_{i=1}^n \mathcal{L}_i \xrightarrow{\phi_n} \pi_n^*\mathcal{S}^+. \tag{5.12}$$

Let $Z_n \subset \mathrm{OG}(5,\mathcal{V})^n$ be the open substack where ϕ_n has rank n. In other words, Z_n is the locus where the n points on $\mathrm{OG}(5,\mathcal{V})$ have independent image under the spinor embedding.

Let $\mathcal{Q}_n$ be the cokernel of $\phi_n|_{Z_n}$, which is a rank $16-n$ vector bundle on Z_n. The fiber of $\mathrm{Gr}(7-n,\mathcal{Q}_n)$ over $(p_1,\ldots,p_n) \in \mathrm{OG}(5,\mathcal{V})^n$ parametrizes linear spaces $\mathbb{P}^6 \subset \mathbb{P}S^+$ that contain the n points p_i. In other words, we can identify $\mathrm{Gr}(7-n,\mathcal{Q}_n)$ with the locally closed substack

$$\{(p_1,\ldots,p_n,\Lambda) : p_i \in \mathbb{P}\Lambda \text{ and } p_i \text{ independent}\} \subset \mathrm{OG}(5,\mathcal{V})^n \times_{\mathrm{BSO}_{10}} \mathrm{Gr}(7,\mathcal{S}^+). \tag{5.13}$$

Lemma 5.13. *For $n \leq 4$, there is an open embedding α_n of $\mathcal{M}^\circ_{7,n}$ in the Grassmann bundle*

$$\begin{tikzcd} \mathcal{M}^\circ_{7,n} \arrow[r, hook, "\alpha_n"] & \mathrm{Gr}(7-n,\mathcal{Q}_n) \arrow[d] & \\ & Z_n \arrow[r, hook] & \mathrm{OG}(5,\mathcal{V})^n. \end{tikzcd}$$

Hence, $\mathcal{M}^\circ_{7,n}$ has the CKgP.

Proof. Let $f\colon \mathcal{C} \to \mathcal{M}^\circ_{7,n}$ be the universal curve and let $\sigma_i\colon \mathcal{M}^\circ_{7,n} \to \mathcal{C}$ be the ith section. The universal version of the Mukai construction furnishes a morphism $\mathcal{C} \to \mathrm{OG}(5,\mathcal{V})$. Precomposing with each of the sections σ_i defines a map $\mathcal{M}^\circ_{7,n} \to \mathrm{OG}(5,\mathcal{V})^n$ over BSO_{10}. We claim that, for $n \leq 4$, the images of $\sigma_1,\ldots,\sigma_n$ must be independent. Indeed, suppose the images of the sections in a fiber, $p_1,\ldots,p_n \in C$, are dependent under the canonical embedding. Then $p_1+\ldots+p_n$ would give rise to a g^1_n on C, but curves in $\mathcal{M}^\circ_{7,n}$ have no g^1_n for $n \leq 4$ by definition.

We also have the map $\mathcal{M}^\circ_{7,n} \to \mathcal{M}^\circ_7 \to \mathrm{Gr}(7,\mathcal{S}^+)$ that sends a curve to its span under the spinor embedding. Taking the product of these maps over BSO_{10} yields a map

$$\mathcal{M}^\circ_{7,n} \to \mathrm{OG}(5,\mathcal{V})^n \times_{\mathrm{BSO}_{10}} \mathrm{Gr}(7,\mathcal{S}^+).$$

This map sends a family of pointed curves $(C,p_1,\ldots,p_n)$ over a scheme T to the data of sections $p_i\colon T \to C \to \mathrm{OG}(5,W^\vee)$ and the subbundle of the spinor representation of $W^\vee$ determined by the span of the fibers of $C \subset \mathrm{OG}(5,W^\vee) \subset \mathbb{P}S^+$ over T. The map evidently factors through the locally closed locus in (5.13), which we identified with $\mathrm{Gr}(7-n,\mathcal{Q}_n)$. In fact, the image is precisely $\mathrm{Gr}(7-n,\mathcal{Q}_n) \smallsetminus \Delta$, where Δ is the closed locus such that $\mathbb{P}\Lambda \cap \mathrm{OG}(5,W^\vee)$ is not a family of smooth curves. Indeed, on the complement of Δ, an inverse map $\mathrm{Gr}(7-n,\mathcal{Q}_n) \smallsetminus \Delta \to \mathcal{M}^\circ_{7,n}$ is defined by sending $(p_1,\ldots,p_n,\Lambda)$ to the curve $C = \mathbb{P}\Lambda \cap \mathrm{OG}(5,W^\vee)$ together with the sections $p_i \in \mathbb{P}\Lambda \cap \mathrm{OG}(5,W^\vee) = C$.

By Corollary 5.11, we know $\mathrm{OG}(5,\mathcal{V})^n$ has the CKgP. To complete the proof, apply Proposition 4.2(1) and (5). □

We now identify the restrictions of the universal bundles on $\mathrm{OG}(5,\mathcal{V})^n$ and $\mathrm{Gr}(7-n,\mathcal{Q}_n)$ to $\mathcal{M}^\circ_{7,n}$ along α_n.

Lemma 5.14. *Let $f\colon \mathcal{C} \to \mathcal{M}^\circ_{7,n}$ be the universal curve and let σ_i denote the image of the ith section. The vector bundle $\alpha_n^*\mathcal{U}_i$ sits in an exact sequence*

$$0 \to \alpha_n^*\mathcal{U}_i \to \mathrm{Sym}^2 f_*(\omega_f(-\sigma_i)) \to f_*((\omega_f(-\sigma_i))^{\otimes 2}) \to 0. \tag{5.14}$$

In particular, all classes pulled back from $\mathrm{OG}(5,\mathcal{V})^n$ to $\mathcal{M}^\circ_{7,n}$ are tautological.

Proof. The composition of α_n with projection onto the ith factor of Z_n is the map that sends a pointed curve $(C, p_1, \ldots, p_n)$ to the image of p_i under the canonical map from C to $\mathrm{OG}(5, W^\vee) \cong \mathrm{OG}(5, 10)$. By construction, the fiber at p_i of the universal rank 5 bundle on $\mathrm{OG}(5, 10)$ is $W_p^\perp$. Equation (5.14) is the relative version of (5.9). By Grothendieck–Riemann–Roch, the middle and right terms in (5.14) have tautological Chern classes. It follows that the Chern classes of $\alpha_n^* \mathcal{U}_i$ are also tautological. The last claim now follows from Corollary 5.11. $\square$

Lemma 5.15. *We have $c_1(\alpha_n^* \mathcal{L}_i) = -\psi_i$ in $A^1(\mathcal{M}_{7,n}^\circ)$.*

Proof. Let $f \colon \mathcal{C} \to \mathcal{M}_{7,n}^\circ$ be the universal curve. The line bundle $\alpha^* \mathcal{L}_i$ is the pullback of $\mathcal{O}_{\mathbb{P}\mathcal{S}^+}(-1)$ along the composition

$$\mathcal{M}_{7,n}^\circ \xrightarrow{\sigma_i} \mathcal{C} \to \mathrm{OG}(5, \mathcal{V}) \xrightarrow{\iota} \mathbb{P}\mathcal{S}^+.$$

But the above composition also factors as

$$\mathcal{M}_{7,n}^\circ \xrightarrow{\sigma_i} \mathcal{C} \xrightarrow{|\omega_f|} \mathbb{P}(f_*\omega_f)^\vee \to \mathbb{P}\mathcal{S}^+,$$

and $\mathcal{O}_{\mathbb{P}\mathcal{S}^+}(-1)$ restricts to $\mathcal{O}_{\mathbb{P}(f_*\omega_f)^\vee}(-1)$. Hence, $\alpha_n^* \mathcal{L}_i$ is $\sigma_i^* \mathcal{O}_{\mathbb{P}(f_*\omega_f)^\vee}(-1) = (\sigma_i^* \omega_f)^\vee$. $\square$

Lemma 5.16. *Let $\mathcal{E}_n$ be the tautological rank $7-n$ subbundle on $\mathrm{Gr}(7-n, \mathcal{Q}_n)$. The pullback $\alpha_n^* \mathcal{E}_n$ sits in an exact sequence*

$$0 \to \bigoplus_{i=1}^n \alpha_n^* \mathcal{L}_i \to (f_*\omega_f)^\vee \to \alpha_n^* \mathcal{E}_n \to 0. \tag{5.15}$$

In particular, the Chern classes of $\alpha_n^* \mathcal{E}_n$ are tautological.

Proof. Recall that we defined $\mathcal{Q}_n$ as the cokernel of (5.12). The map onto the second factor in (5.13), $p \colon \mathrm{Gr}(7-n, \mathcal{Q}_n) \to \mathrm{Gr}(7, \mathcal{S}^+)$, is defined by sending a $7-n$ dimensional subspace to its preimage under the quotient map $\mathcal{S}^+ \to \mathcal{Q}_n$. Hence, there is an exact sequence on $\mathrm{Gr}(7-n, \mathcal{Q}_n)$

$$0 \to \bigoplus_{i=1}^n \mathcal{L}_i \to p^* \mathcal{E} \to \mathcal{E}_n \to 0.$$

Then note that $\alpha_n^* p^* \mathcal{E}$ is the same as the pullback of $\mathcal{E}$ along $\mathcal{M}_{7,n}^\circ \to \mathcal{M}_7^\circ \to \mathrm{Gr}(7, \mathcal{S}^+)$, which is the dual of the Hodge bundle. The last claim follows by combining the exact sequence (5.15) with Lemma 5.15. $\square$

Lemma 5.17. *The Chow ring of $\mathcal{M}_{7,n}^\circ$ is generated by tautological classes.*

Proof. By Lemma 5.13, we know that $\mathcal{M}_{7,n}^\circ$ is an open substack of $\mathrm{Gr}(7-n, \mathcal{Q}_n)$. The Chow ring of the latter is generated by pullbacks of classes from $\mathrm{OG}(5, \mathcal{V})^n$ and by the Chern classes of the tautological subbundle $\mathcal{E}_n$. By Lemmas 5.14 and 5.16 respectively, both of these classes restrict to tautological classes on $\mathcal{M}_{7,n}^\circ$. $\square$

6. Applications to even cohomology

Here we use the results from Sections 3, 4, and 5 to prove Theorem 1.5.

6.1. **The degree 4 cohomology of $\overline{\mathcal{M}}_{g,n}$ is tautological.**

Proof of Theorem 1.5(1). By [1], we know that $H^k(\overline{\mathcal{M}}_{g,n})$ is tautological for $k \leq 3$. Therefore, using Theorem 1.3(1), we deduce that $H^4(\overline{\mathcal{M}}_{g,n})$ is contained the STE generated by $H^4(\overline{\mathcal{M}}_{g',n'})$ for $g' < 7$ and $n' \leq 4$. By Proposition 4.5, $W_4H^4(\mathcal{M}_{g',n'})$ is tautological for g' and n' in this range, and it follows that $H^4(\overline{\mathcal{M}}_{g,n})$ is tautological. □

Proof of Theorem 1.5(2). By Theorem 1.5(1) and [1], all classes in $H^6(\overline{\mathcal{M}}_{g,n})$ that are pushed forward from the boundary are tautological. Also, $H^6(\mathcal{M}_{g,n})$ is stable and hence tautological for $g \geq 10$ [32]. It follows that $H^6(\overline{\mathcal{M}}_{g,n})$ is tautological for $g \geq 10$. □

Remark 6.1. By Lemma 3.1, to show that $H^6(\overline{\mathcal{M}}_{g,n})$ is tautological for all g and n, it would suffice to show this for $g \leq 9$ and $n \leq 6$. In principle, this can be checked computationally as follows. By Theorem 1.5(3), we know that $H_6(\overline{\mathcal{M}}_{g,n})$ is tautological for all g and n. Thus, if the intersection pairing

$$RH^6(\overline{\mathcal{M}}_{g,n}) \times RH_6(\overline{\mathcal{M}}_{g,n}) \to \mathbb{Q}$$

is perfect, then $H^6(\overline{\mathcal{M}}_{g,n})$ is tautological. In principle, one could compute this pairing using the Sage package admcycles [11]. In practice, however, this computation is too memory intensive to carry out.

6.2. **The low degree homology of $\overline{\mathcal{M}}_{g,n}$ is tautological.** Let H_k denote Borel–Moore homology with coefficients in $\mathbb{Q}$ or $\mathbb{Q}_\ell$, together with its mixed Hodge structure or Galois action. By Poincaré duality, we have the identification of vector spaces

$$H^{2d_{g,n}-k}(\mathcal{M}_{g,n}) \cong H_k(\mathcal{M}_{g,n}).$$

From the long exact sequence in Borel–Moore homology, we have a right exact sequence

$$H_k(\widetilde{\partial\mathcal{M}_{g,n}}) \to H_k(\overline{\mathcal{M}}_{g,n}) \to W_{-k}H_k(\mathcal{M}_{g,n}) \to 0 \tag{6.1}$$

for all k.

Remark 6.2. One can prove analogues of Theorem 1.3 and Lemma 2.1 for homology. In particular, for each k, there is a finitely generated STE that contains $H_k(\overline{\mathcal{M}}_{g,n})$ for all g and n, with explicit bounds on the generators.

Proof of Theorem 1.5(3). When $g = 0$, all cohomology is tautological [16]. By [4, Proposition 2.1] and the universal coefficient theorem, for $g \geq 1$ we have:

$$H_k(\mathcal{M}_{g,n}) = 0 \text{ for } \begin{cases} k < 2g \text{ and } n = 0, 1; \\ k < 2g - 2 + n \text{ and } n \geq 2. \end{cases}$$

Combining this with the exact sequence (6.1), we reduce inductively to the finitely many cases $k \geq 2g$ and $n = 0, 1$ or $k \geq 2g - 2 + n$ and $n \geq 2$. When $g \geq 3$, in all of these cases we know that $\mathcal{M}_{g,n}$ has the CKgP and $A^*(\mathcal{M}_{g,n}) = R^*(\mathcal{M}_{g,n})$ (see Table 1). By Proposition 4.5, it follows that $W_{-k}H_k(\mathcal{M}_{g,n})$ is tautological, so again by induction and the exact sequence (6.1) we reduce to the cases of $g = 1, 2$. When $g = 1$, all even cohomology is tautological [24]. When $g = 2$, we apply Corollary 2.6. □

Corollary 6.3. *As Hodge structures or Galois representations, we have*

$$H^{2d_{g,n}-14}(\overline{\mathcal{M}}_{g,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{L}^{d_{g,n}-7}$$

Proof. We proved $H_{14}(\overline{\mathcal{M}}_{g,n})$ is tautological and hence algebraic. Applying Poincaré duality yields the corollary. $\square$

7. The thirteenth and fifteenth homology of $\overline{\mathcal{M}}_{g,n}$

In this section, we prove Theorems 1.6 and 1.7. As a corollary, we obtain restrictions on the Hodge structure and Galois representations appearing in $H_{13}(\overline{\mathcal{M}}_{g,n})$ and $H_{15}(\overline{\mathcal{M}}_{g,n})$.

As usual, we argue by induction on g and n. First, we must treat one extra base case.

Lemma 7.1. *The group $H^{13}(\overline{\mathcal{M}}_{2,11})$ is in the STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$.*

Proof. In genus 2, Bergström and Faber have an implementation of the Getzler–Kapranov formula to determine $H^*(\overline{\mathcal{M}}_{2,n})$. In particular, they show that $H^{13}(\overline{\mathcal{M}}_{2,11})$ consists of 22 copies of LS_{12} [2]. For each $i = 1, \dots, 11$ consider the maps

$$\alpha_i\colon H^{11}(\overline{\mathcal{M}}_{1,11}) \cong H^{11}(\overline{\mathcal{M}}_{1,\{i\}^c\cup p} \times \overline{\mathcal{M}}_{1,\{i,p'\}}) \to H^{13}(\overline{\mathcal{M}}_{2,11})$$

and

$$\beta_i\colon H^{11}(\overline{\mathcal{M}}_{1,11}) \xrightarrow{\pi_i^*} H^{11}(\overline{\mathcal{M}}_{1,12} \times \overline{\mathcal{M}}_{1,1}) \to H^{13}(\overline{\mathcal{M}}_{2,11}).$$

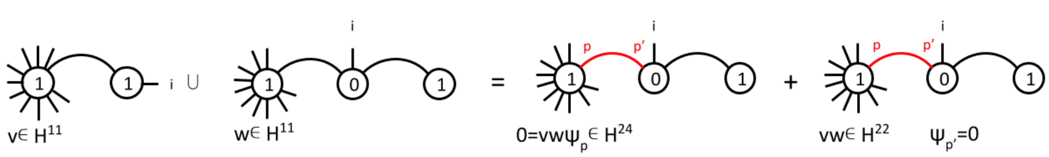
It suffices to show that these 22 maps in $\mathrm{Hom}(\mathsf{LS}_{12}, H^{13}(\overline{\mathcal{M}}_{2,11}))$ are independent. To see this, we think of the classes in the images of α_i and β_i as decorated 1-edge graphs where the decoration on the first genus 1 vertex is a class in H^{11} of that vertex. We then intersect with classes in the complementary degree. Our complementary degree classes lie in the images of

$$a_i\colon H^{11}(\overline{\mathcal{M}}_{1,11}) \cong H^{11}(\overline{\mathcal{M}}_{1,\{i\}^c\cup p} \times \overline{\mathcal{M}}_{0,\{p',i,q\}} \times \overline{\mathcal{M}}_{1,\{q'\}}) \to H^{15}(\overline{\mathcal{M}}_{2,11})$$

and

$$b_i\colon H^{11}(\overline{\mathcal{M}}_{1,11}) \xrightarrow{\pi_i^*} H^{11}(\overline{\mathcal{M}}_{1,12} \times \overline{\mathcal{M}}_{0,\{p',q,q'\}}) \to H^{15}(\overline{\mathcal{M}}_{2,11}).$$

We again think of the image classes as decorated 2-edge graphs where the decorations live in H^{11}. We claim that the intersection of the image of α_i with the image of any b_j vanishes. To see this, intersect decorated graphs as in [15, Appendix A]. The rules for pulling back decorations in H^{11} to boundary strata are determined by [8, Lemma 2.2]. For example, if $i \neq j$, the boundary strata corresponding to the graphs do not meet, while if $i = j$, then we obtain the following decorated graphs



The excess edges are pictured in red. Each term on the right hand side vanishes for degree reasons. We similarly compute that the intersection of the image of α_i with the image of a_j vanishes unless $i = j$. If $i = j$, then we obtain a full rank 2×2 matrix, which is the pairing matrix for $H^{11}(\overline{\mathcal{M}}_{1,11}) \times H^{11}(\overline{\mathcal{M}}_{1,11}) \to \mathbb{Q}$ with respect to a suitable basis. Similarly, the pairing of the image of β_i with the image of b_j vanishes unless $i = j$, in which case we obtain a full rank 2×2 matrix. The resulting matrix is block upper triangular with full rank diagonal blocks, so the union of $\{\alpha_i\}$ and $\{\beta_i\}$ is independent. $\square$

Proof of Theorem 1.6. The proof is similar to that of Theorem 1.5(3). When $g = 0$, all odd cohomology vanishes [16], so we can assume $g \geq 1$. By [4, Proposition 2.1] and the universal coefficient theorem, we have that for $g \geq 1$

$$H_{13}(\mathcal{M}_{g,n}) = 0 \text{ for } \begin{cases} 13 < 2g \text{ and } n = 0, 1 \\ 13 < 2g - 2 + n \text{ and } n \geq 2. \end{cases}$$

Combining this with the exact sequence (6.1), we reduce inductively to the finitely many cases where $13 \geq 2g$ and $n = 0, 1$ or $13 \geq 2g - 2 + n$ and $n \geq 2$. All such cases except for (g, n) in $Z = \{(2, 11), (1, 11), (1, 12), (1, 13)\}$ have the CKgP (see Table 1). By Proposition 4.5, it follows that $W_{-13}H_{13}(\mathcal{M}_{g,n}) = 0$ for $(g, n) \notin Z$. By induction and the exact sequence (6.1), we reduce to the exceptional cases of $(g, n) \in Z$. The case $(g, n) = (2, 11)$ follows from Lemma 7.1 and Hard Lefschetz. The cases where $g = 1$ follow from Proposition 2.2 and induction on n. □

Corollary 7.2. *As Hodge structures or Galois representations, we have*

$$H^{2d_{g,n}-13}(\overline{\mathcal{M}}_{g,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{L}^{d_{g,n}-12}\mathsf{S}_{12}$$

for all g and n.

Proof. Let $S^* = S^*_\omega$ be the STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$. By Theorems 1.5(3) and 1.6, when $k \leq 13$, we have $H_k(\overline{\mathcal{M}}_{g,n}) = S_k(\overline{\mathcal{M}}_{g,n})$. The result clearly holds for

$$H_{13}(\overline{\mathcal{M}}_{1,11}) \cong H^9(\overline{\mathcal{M}}_{1,11}) = 0.$$

By definition, $S_{13}(\overline{\mathcal{M}}_{g,n})$ is generated by the images of the pushforward map

$$S_{13}(\widetilde{\partial\overline{\mathcal{M}}_{g,n}}) \to S_{13}(\overline{\mathcal{M}}_{g,n})$$

and the pullback maps

$$S_{11}(\overline{\mathcal{M}}_{g,n-1}) \to S_{13}(\overline{\mathcal{M}}_{g,n}).$$

By induction on g and n, it follows that $S_{13}(\overline{\mathcal{M}}_{g,n}) = H_{13}(\overline{\mathcal{M}}_{g,n})$ is isomorphic to a sum of Tate twists of S_{12}, up to semisimplification. Applying Poincaré duality yields the result. □

Remark 7.3. Theorem 1.6 says that $H_{13}(\overline{\mathcal{M}}_{g,n})$ is contained in the STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$. Roughly speaking, this means that it has a graphical presentations, in which the generators are graphs of the sort appearing in the proof of Lemma 7.1. In forthcoming work, we will show that $H^{13}(\overline{\mathcal{M}}_{g,n})$ is also contained in this STE and provide a complete list of relations among the corresponding decorated graph generators.

Proof of Theorem 1.7. The proof is similar to that of Theorem 1.6. When $g = 0$, all odd cohomology vanishes [16], so we can assume $g \geq 1$. By [4, Proposition 2.1] and the universal coefficient theorem, we have that for $g \geq 1$

$$H_{15}(\mathcal{M}_{g,n}) = 0 \text{ for } \begin{cases} 15 < 2g \text{ and } n = 0, 1 \\ 15 < 2g - 2 + n \text{ and } n \geq 2. \end{cases}$$

Combining this with the exact sequence (6.1), we reduce inductively to the finitely many cases $15 \geq 2g$ and $n = 0, 1$ or $15 \geq 2g - 2 + n$ and $n \geq 2$. In such cases when $g \geq 3$, $\mathcal{M}_{g,n}$ has the CKgP, and so by induction and the exact sequence (6.1), we reduce to the cases where $g = 1$ and $11 \leq n \leq 14$, or $g = 2$ and $n = 11$. When $g = 2$ and $n = 11$, the result follows from Hard Lefschetz and Lemma 7.1. For the genus 1 cases, the claim follows from Proposition 2.2. $\square$

Corollary 7.4. *As Hodge structures or Galois representations, we have*

$$H^{2d_{g,n}-15}(\overline{\mathcal{M}}_{g,n})^{\mathrm{ss}} \cong \bigoplus \mathsf{L}^{d_{g,n}-13}\mathsf{S}_{12} \oplus \bigoplus \mathsf{L}^{d_{g,n}-15}\mathsf{S}_{16}$$

for all g and n.

Proof. The proof is similar to that of Corollary 7.2, using that $H^{15}(\overline{\mathcal{M}}_{1,15}) \cong \mathsf{S}_{16}$. $\square$

We expect that Theorem 1.7 can be improved as follows.

Conjecture 7.5. *The STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$ contains $H^{15}(\overline{\mathcal{M}}_{g,n})$ for $g \geq 2$.*

The proof of Theorem 1.1 in the next section shows that for $g \geq 2$, $H^{15}(\overline{\mathcal{M}}_{g,n})$ contains no copies of S_{16}. Conjecture 7.5 is therefore equivalent to the assertion that the STE generated by $H^{11}(\overline{\mathcal{M}}_{1,11})$ and $H^{15}(\overline{\mathcal{M}}_{1,15})$ contains $H^{15}(\overline{\mathcal{M}}_{g,n})$ for all g and n.

8. Proof of Theorem 1.1

The first two statements of Theorem 1.1 follow immediately from Corollaries 6.3 and 7.2 by inverting copies of L. In Corollary 7.4, we can invert L to see that $H^{15}(\overline{\mathcal{M}}_{g,n})^{\mathrm{ss}}$ is a direct sum of copies of $\mathsf{L}^2\mathsf{S}_{12}$ and S_{16}. To finish, it suffices to show that when $g \geq 2$, no copies of S_{16} appear. When $g = 2$, there are no copies of S_{16} in $H^{15}(\overline{\mathcal{M}}_{2,n})^{\mathrm{ss}}$ by Corollary 2.8.

To show this vanishing when $g \geq 3$, we will use a similar strategy to that of [8]. We study the first two maps in the weight 15 complex:

$$H^{15}(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\alpha} \bigoplus_{|E(\Gamma)|=1} H^{15}(\overline{\mathcal{M}}_{\Gamma})^{\mathrm{Aut}(\Gamma)} \to \bigoplus_{|E(\Gamma)|=2} \left(H^{15}(\overline{\mathcal{M}}_{\Gamma}) \otimes \det E(\Gamma)\right)^{\mathrm{Aut}(\Gamma)}. \tag{8.1}$$

Here, Γ is a stable graph of genus g with n legs. The first map is the pullback to the normalization of the boundary $\widetilde{\partial \mathcal{M}_{g,n}}$. The second map is explicitly described in [8, Section 4.1].

Lemma 8.1. *If $g \geq 3$, then the pullback map α in* (8.1) *is injective.*

Proof. The map α is injective when $\mathrm{gr}^W_{15} H^{15}_c(\mathcal{M}_{g,n}) = 0$. By [4, Proposition 2.1], this holds whenever $2g - 2 + n > 15$. When $2g - 2 + n \leq 15$, $\mathcal{M}_{g,n}$ has the CKgP (see Table 1). The result follows from Proposition 4.5. $\square$

We now assume $g \geq 3$, and we study the second map in (8.1) as a morphism of Hodge structures. In particular, we consider the $H^{15,0}$ part:

$$\bigoplus_{|E(\Gamma)|=1} H^{15,0}(\overline{\mathcal{M}}_{\Gamma})^{\mathrm{Aut}(\Gamma)} \xrightarrow{\beta} \bigoplus_{|E(\Gamma)|=2} \left(H^{15,0}(\overline{\mathcal{M}}_{\Gamma}) \otimes \det E(\Gamma)\right)^{\mathrm{Aut}(\Gamma)}. \tag{8.2}$$

By Lemma 8.1, to prove the theorem in the category of Hodge structures, it suffices to show that β is injective. The domain is a direct sum over graphs with one edge. If Γ has one vertex and a loop, then $H^{15,0}(\overline{\mathcal{M}}_{\Gamma}) = H^{15,0}(\overline{\mathcal{M}}_{g-1,n+2}) = 0$ by induction on g. If Γ has two vertices with an edge connecting them, then by the Künneth formula, we have

$$H^{15,0}(\overline{\mathcal{M}}_{\Gamma}) = H^{15,0}(\overline{\mathcal{M}}_{a,A\cup p}) \oplus H^{15,0}(\overline{\mathcal{M}}_{b,A^c\cup q}).$$

Assume $a \geq b$. By induction, the above is non-vanishing only if $b = 1$ and $|A^c| \geq 14$. Because $g \geq 3$, it follows that $H^{15,0}(\overline{\mathcal{M}}_\Gamma) = H^{15,0}(\overline{\mathcal{M}}_{1,A^c \cup q})$ in this case. Let Γ' be the graph obtained from Γ by adding a loop on the genus $g-1$ vertex and decreasing the genus accordingly, as in Figure 2.

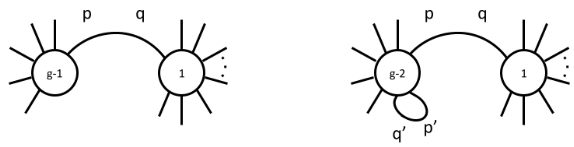


FIGURE 2. The graph Γ on the left and Γ' on the right.

Then $H^{15,0}(\overline{\mathcal{M}}_{\Gamma'})$ has a summand $H^{15,0}(\overline{\mathcal{M}}_{1,A^c \cup q})$, and the map $H^{15,0}(\overline{\mathcal{M}}_\Gamma) \to H^{15,0}(\overline{\mathcal{M}}_{\Gamma'})$ injects into that summand.

The proof in the category of Galois representations is similar to that for Hodge structures. One applies $\mathrm{Hom}(\mathsf{S}_{16}, -)$ to the second map in (8.1) and argues by induction on g and n to see that this map is injective. □

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