SYMONS SYMPOSIUM TALK: METRIZATION OF DIFFERENTIAL PLURIFORMS ON BERKOVICH ANALYTIC SPACES

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1. Introduction

Kontsevich and Soibelman introduced in [KS06] a metric on the canonical sheaf of a $\mathbf{C}((t))$-analytic Calabi-Yau manifold, and their definition was extended by Mustaţă and Nicaise in [MN13] to pluricanonical sheaves $\omega_X^{\otimes m}$ on a smooth algebraizable Berkovich space X over a discretely valued field k. Their definition associates to a pluricanonical form $\phi \in \Gamma(\omega_X^{\otimes m})$ a real-valued weight function wt_ϕ defined on a subset of X. First, one defines wt_ϕ at divisorial points of X and then extends it by continuity to PL subspaces (or skeletons) of X.

This lecture discusses a recent progress made in [Tem14]. The main goal of that work was to metrize the sheaves $\omega_X^{\otimes m}$ in a purely analytic way, thereby eliminating unnecessary technical assumptions. In particular, this method applies to all rig-smooth spaces over an arbitrary non-archimedean field, including the trivially valued ones, and deals with all points on an equal footing, thereby providing a norm function $\|\phi\|\colon X \to \mathbf{R}_+$. Moreover, for any morphism of k-analytic spaces $X \to S$ we naturally metrize $\Omega_{X/S}$ and all related sheaves, such as $S^n\Omega_{X/S}$, $\bigwedge^m \Omega_{X/S}$, etc. In particular, this applies to families of spaces parameterized by S.

2. [KS06], [MN13] and the weight function

The main idea of defining wt_ϕ is as follows. For a divisorial point $x \in X$ and $\phi \in \Gamma(\omega_X)$ find an algebraic normal k°-model $\mathcal{X}$ such that $\eta = \pi_{\mathcal{X}}(x)$ is a generic point of the closed fiber $\mathcal{X}_s$. The regular locus $U \subseteq \mathcal{X}$ contains η and the relative canonical sheaf ω_{U/k° is invertible, hence the rational section ϕ of ω_{U/k° defines a vanishing divisor E on U. It is easy to see that locally at η we have an equality $nE = (\pi)$, where $n > 0$ and $\pi \in k^\times$, and one sets $\mathrm{wt}_\phi(x) = |\pi_k| \cdot |\pi|^{1/n}$, where π_k is a uniformizer of k. Independence of choices follows by easy computations with lci morphisms between different models. This is the main source of restrictions of the method, such as assuming k discretely valued and X algebraizable.

Remark 2.1. (i) This agrees with the definition of [MN13] with the only difference that we switched to multiplicative notation.

(ii) The definition in [KS06] operates with $k = \mathbf{C}((t))$ and uses the order of vanishing of $\phi \wedge \frac{dt}{t}$ in $\omega_{\mathcal{X}/\mathbf{C}}$, but it misses a $+1$ summand to compensate for the pole of $\frac{dt}{t}$.

Theorem 2.2 (Mustaţă-Nicaise)**.** *(i)* wt_ϕ *extends by continuity to any PL subspace of* X*.*

(ii) If X *possesses a semi-stable model then* wt_ϕ *extends to an upper semicontinuous function on the whole* X*.*

Remark 2.3. The main application of this theorem in [MN13] is a natural construction of skeletons of analytic spaces.

3. Main idea and possible realizations

3.1. **Fiberwise metric.** Assume that X is rig-smooth of pure dimension d. For each $x \in X$ one can show that the completed fiber $\omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{H}(x)$ coincides with the vector space $\bigwedge^d \widehat{\Omega}_{\mathcal{H}(x)/k}$. In first approximation, the idea of defining a generalized weight function is to provide the latter space with a metric $\| \ \|_x$ by taking the image of $\bigwedge^d \widehat{\Omega}_{\mathcal{H}(x)^\circ/k^\circ}$ as the unit ball and to set $\mathrm{wt}_\phi(x) = \|\phi(x)\|_x$.

Example 3.1.1. (i) Let us consider the simplest one-dimensional example when $X = \mathcal{M}(k\{t\})$ is a unit disc over an algebraically closed k. If x is its maximal point then dt generates $\widehat{\Omega}_{\mathcal{H}(x)^\circ/k^\circ}$ and hence $\|dt\|_x = 1$. In general, $\widehat{\Omega}_{\mathcal{H}(x)^\circ/k^\circ}$ is generated by the elements $\frac{d(t-a)}{c} = \frac{dt}{c}$ such that $|\frac{t-a}{c}|_x \le 1$ and hence $\|dt\|_x = \inf_{a\in k} |t-a|_x$. Thus, $\|dt\|$ coincides with the radius function $r(x)$ on X. In particular, it is upper semicontinuous but not continuous.

(ii) If x is a type 3 point corresponding to a generalized Gauss norm then $|\frac{dt}{t}|_x = 1$ but $\frac{dt}{t} \notin \widehat{\Omega}_{\mathcal{H}(x)^\circ/k^\circ}$. In particular, $\widehat{\Omega}_{\mathcal{H}(x)^\circ/k^\circ}$ is not the closed unit ball of the metric it defines. In fact, it is the open unit ball in this case.

Remark 3.1.2. The example with type 3 point indicates that it is better to define the norm using $\widehat{\Omega}^{\log}_{\mathcal{H}(x)^\circ/k^\circ} := \widehat{\Omega}^{\log}_{(\mathcal{H}(x)^\circ,\mathcal{H}(x)^\circ\setminus\{0\})/(k^\circ,k^\circ\setminus\{0\})}$. This module contains $\widehat{\Omega}_{\mathcal{H}(x)^\circ/k^\circ}$ and the quotient is annihilated by any $\pi \in k$ with $|\pi| < 1$. In particular, both define the same norm if the valuation on k is not discrete. We will see that in the discrete case one should use the logarithmic module.

More generally, for any morphism $f\colon X \to S$ with $x \in X$ and $s = f(x)$ one can define a seminorm $\| \ \|_\Omega$ on $\widehat{\Omega}_{\mathcal{H}(x)/\mathcal{H}(s)}$ by taking the image of $\widehat{\Omega}^{\log}_{\mathcal{H}(x)^\circ/\mathcal{H}(s)^\circ}$ as the unit ball, and $\| \ \|_\Omega$ induces natural seminorms on the modules $S^n\widehat{\Omega}_{\mathcal{H}(x)/\mathcal{H}(s)}$, $\bigwedge S^n\widehat{\Omega}_{\mathcal{H}(x)/\mathcal{H}(s)}$, etc. These fiberwise norms induce *Kähler metrics* on the sheaves $S^n\Omega_{X/S}$, $\bigwedge^m \Omega_{X/S}$, etc. Having described a general idea, let us discuss more formally a few possibilities to metrize sheaves. For concreteness, we consider a sheaf of rings $\mathcal{A}$ on a topological space X. Once $\mathcal{A}$ is metrized, sheaves of $\mathcal{A}$-modules are dealt with similarly.

3.2. **First approach.** If $\mathcal{A}$ is a sheaf of k-algebras, one can metrize it by providing the unit ball $\mathcal{A}^\diamond$, which is a subsheaf of k°-modules such that $\mathcal{A} = \mathcal{A}^\diamond \otimes_{k^\circ} k$. In this approach, the Kähler metric $\| \ \|_{\Omega,X/S}$ on $\Omega_{X/S}$ corresponds to $\mathcal{O}^\circ_X d_{X/S}(\mathcal{O}^\circ_X)$, the minimal $\mathcal{O}^\circ_X$-submodule generated by the image of $\mathcal{O}^\circ_X$ under $d_{X/S}$.

Drawbacks: The generality is reduced to k-algebras. The case of a discretely valued k is problematic, and the case of a trivially valued k does not make sense. (Can be by-passed by extending k from the beginning.)

3.3. **Second approach.** Define metrics on $\mathcal{A}$ by metrizing the stalks $\mathcal{A}_x$. In this approach, $\| \ \|_{\Omega,X/S}$ is the minimal seminorm on $\Omega_{X/S}$ such that the differential $d_{X/S}$ is a non-expanding homomorphism of seminormed sheaves.

Drawbacks: This approach is used in the paper, but it is not topos-theoretic. In particular, in order to apply it to a non-good Berkovich space X one cannot work with the G-topological space X_G, and has to switch to its underlying topological space $|X_G|$. In particular, this requires to study points of X_G.

3.4. **Third approach.** Define sheaves of seminormed rings and modules as sheaves with values in the (non-expanding) categories of seminormed rings/modules.

Drawbacks: need to develop enough category theory of seminormed modules. Also, has to extend seminorms so that they can obtain the infinite value, since sections on open non-compact sets can be unbounded.

3.5. **Seminormed algebra.** Finally, let us discuss the algebraic side of the picture. Analytic geometry mainly uses Banach rings and modules, but seminormed rings and modules often appear in intermediate constructions, e.g. in the definition of $\widehat{\otimes}$. Even more importantly, the stalks $\mathcal{O}_{X,x}$ are just seminormed rings. For this reason, [Tem14] is based on the "seminormed algebra" whose main objects are seminormed rings and modules. In addition, it is important to consider only non-expanding homomorphisms, although one usually works with all bounded homomorphisms between Banach modules.

4. Seminormed sheaves

Formal definitions of the second and third approaches are as follows. For concreteness, we consider sheaves of rings.

Definition 4.1. Let $\mathcal{A}$ be a sheaf of rings on a topological space X. A *seminorm* $|\ |$ on $\mathcal{A}$ is a family of seminorms $|\ |_x$ on the stalks $\mathcal{A}_x$ such that the following sheaf condition holds: for any section $s \in \mathcal{A}(U)$ the function $|s|\colon U \to \mathbf{R}_+$ is upper semicontinuous.

Definition 4.2. A *sheaf of seminormed rings* $(\mathcal{A}, |\ |)$ consists of a sheaf of rings $\mathcal{A}$ and a quasi-norm $|\ |_U\colon \mathcal{A}(U) \to \mathbf{R}_+ \cup \{\infty\}$ (i.e. a seminorm that may take an infinite value too) for any open $U \subseteq X$ such that the following conditions hold:

- Boundedness: for any U and $s \in \mathcal{A}(U)$ there exists a covering $U = \cup_i U_i$ such that $|s|_{U_i} < \infty$ for each i.
- The sheaf condition: If $U = \cup_i U_i$ then $|s|_U = \sup_i |s|_{U_i}$.

The two definitions are equivalent: given $\{|\ |_x\}_{x\in X}$ one defines $|s|_U = \sup_{x\in U} |s|_x$ and given $\{|\ |_U\}_{U\subseteq X}$ one defines $|s|_x = \inf_{x\in U} |s|_U$. It is easy to see that these constructions are inverse one to another. Also, one can naturally extend operations on sheaves of modules, such as $\otimes$, $\bigwedge^n$, etc. to operations on seminormed sheaves of modules. For example, $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is provided with the maximal seminorm such that the bilinear map $M \times N \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is non-expanding.

5. Kähler seminorms for seminormed rings

Definition 5.1. Let $A \to B$ be a non-expanding homomorphism of seminormed rings. The Kähler seminorm $\|\ \|_\Omega$ on $\Omega_{B/A}$ is the maximal seminorm making the A-homomorphism $d\colon B \to \Omega_{B/A}$ non-expanding.

In fact, $(\Omega_{B/A}, \|\ \|_\Omega)$ provides a natural extension of the theory of Kähler differentials to the seminormed setting.

Lemma 5.2. *(i) $d_{B/A}$ is the universal contracting A-derivation of B with values in a seminormed B-module:*

$$\mathrm{Hom}_{B,\mathrm{nonexp}}(\Omega_{B/A},M)\widetilde{\to}\mathrm{Der}_{A,\mathrm{nonexp}}(B,M)$$

(ii) If I is the kernel of the homomorphism of seminormed rings $B\otimes_A B\to B$ then the classical isomorphism $\phi\colon\Omega_{B/A}\widetilde{\to}I/I^2$ is an isometry.

(iii) If $A\to A'$ is a non-expanding homomorphism and $B'=B\otimes_A A'$ then the classical isomorphism $\Omega_{B/A}\otimes_B B'\widetilde{\to}\Omega_{B'/A'}$ is an isometry.

Remark 5.3. If $A\to B\to C$ is a sequence of non-expanding homomorphisms then the first fundamental sequence

$$\Omega_{B/A}\otimes_B C\stackrel{\psi_{C/B/A}}{\longrightarrow}\Omega_{C/A}\stackrel{\phi}{\to}\Omega_{C/B}\to 0$$

does not have to be strictly admissible in the middle, i.e. the norm on $\mathrm{Im}(\psi_{C/B/A})$ can be larger than the norm on $\mathrm{Ker}(\phi)$. In case of valued fields one can control this non-admissibility by use of ramification theory, and this will be important later.

6. Kähler seminorms for real-valued fields

If K is a real-valued field and $\phi\colon A\to K$ is a homomorphism we set $A^\circ=\phi^{-1}(K^\circ)$ and $\Omega^{\log}_{K^\circ/A^\circ}=\Omega^{\log}_{(K^\circ,K^\circ\setminus\{0\})/(A^\circ,A^\circ\setminus\{0\})}$. The following simple theorem expresses the Kähler seminorm in terms of the module of logarithmic differentials.

Theorem 6.1. *Consider the natural homomorphism $h\colon\Omega^{\log}_{K^\circ/A^\circ}\to\Omega_{K/A}$. Then $\mathrm{Im}(h)=\Omega^{\log}_{K^\circ/A^\circ}/\mathrm{torsion}$ and the Kähler seminorm is the maximal K-seminorm on $\Omega_{K/A}$ such that $\|\mathrm{Im}(h)\|_\Omega\le 1$.*

Using this theorem, one can study the non-admissibility of the first fundamental sequence. We will be mainly interested in studying $\psi_{L/K/A}$, where L/K is an extension of real-valued fields. We say that a separable algebraic extension L/K is *almost tame* if the torsion module $\Omega^{\log}_{L^\circ/K^\circ}$ vanishes.

Theorem 6.2. *Assume that L/K is finite and separable, so that $\psi_{L/K/A}$ is an isomorphism. Assume, also, that $\Omega^{\log}_{L^\circ/A^\circ}$ is torsion free (e.g. this happens when A° is an algebraically closed real-valued field). Then $\psi_{L/K/A}$ is an isometry if and only if L/K is almost tame.*

Remark 6.3. In fact, the ratio of the volumes of the unit balls equals to the logarithmic different of L/K.

The following theorem is pretty subtle in the case when L/K is not separable and hence $\psi_{L/K/A}$ is not injective.

Theorem 6.4. *If L/K is an extension of real-valued fields such that K is dense in L then $\psi_{L/K/A}$ is an isometry with a dense image.*

Corollary 6.5. *If K is dense in L then $\widehat{\Omega}_{K/A}=\widehat{\Omega}_{L/A}$, where the completions are taken with respect to the Kähler seminorms.*

7. Kähler seminorm on $\Omega_{X/S}$

Let $f\colon X \to S$ be a morphism of Berkovich spaces. If $x \in X$ and $s = f(x)$ then $\Omega_{X/S,x} = \mathrm{colim}_i \widehat{\Omega}_{\mathcal{B}_i/\mathcal{A}}$ where $\mathcal{M}(\mathcal{A})$ is an affinoid domain containing s and $\mathcal{M}(\mathcal{B}_i)$ run over affinoid domains over $\mathcal{M}(\mathcal{A})$ that contain x. We define a seminorm $\| \ \|_x$ on $\Omega_{X/S,x}$ as the colimit of the Kähler norms on $\widehat{\Omega}_{\mathcal{B}_i/\mathcal{A}}$.

Theorem 7.1. *(i) The completion of $(\Omega_{X/S,x}, \| \ \|_x)$ is naturally isomorphic as a seminormed module to $\widehat{\Omega}_{\mathcal{H}(x)/\mathcal{H}(s)}$.*

(ii) For any section $s \in \Omega_{X/S}(U)$ the function $\|s\|$ is upper semicontinuous. In particular, the stalk seminorms $\| \ \|_x$ give rise to a seminorm $\| \ \|_\Omega$ on $\Omega_{X/S}$, that we call the Kähler seminorm.

(iii) Each function $\|s\|\colon |U_G| \to \mathbf{R}_+$ factors through U. In particular, it is determined by its values on the usual anlytic points.

The main task is to prove (i), and the crucial point here is that if the spaces are good then $\widehat{\Omega}_{\kappa(x)/\kappa(s)} = \widehat{\Omega}_{\mathcal{H}(x)/\mathcal{H}(s)}$ by Corollary 6.5. Once the Kähler seminorm is defined we automatically obtain seminorms on the related sheaves, such as $S^n\Omega_{X/S}$ and $\bigwedge^m \Omega_{X/S}$. We call them Kähler seminorms too.

Example 7.2. Let $X = \mathcal{M}(k\{t\})$ be a unit disc.

(0) Assume that k is algebraically closed. It follows from Example 3.1.1 that the maximality locus of dt on X is the maximal point, and the maximality locus of $\frac{dt}{t}$ on the punched unit disc is its skeleton (i.e. the set of all generalized Gauss points). More generally, the maximality locus of $\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_l}{t_l}$ on a torus $(G_m)^l$ is the skeleton $\mathbf{R}_+^l$.

(i) If k is perfect then $\| \ \|_x = 0$ at any rigid point $x \in X$ since $\Omega_{\mathcal{H}(x)/k} = 0$.

(ii) If k is not perfect, $l = k(a^{1/p})$ is inseparable over k and $l = \mathcal{H}(x)$ for a rigid point $x \in X$, then $\|dt\|_x > 0$. Moreover, one can show that if $\widetilde{k}$ is not perfect then the maximality locus of $\|dt\|$ is an infinite tree.

8. Main results

8.1. **Pullbacks.** We say that a seminormed k-algebra $\mathcal{A}$ is *universally spectral* if for any extension of real-valued fields l/k the product seminorm on $\mathcal{A} \otimes_k l$ is power-multiplicative. One can show that a finite extension K/k is universally spectral if and only if it is almost tame, although this is only checked for tame extensions in [Tem14].

Theorem 8.1.1. *Let $f\colon X \to S$ and $g\colon S' \to S$ be morphisms of Berkovich spaces and $S' = S \times_X X'$. Then $\| \ \|_{\Omega, X'/S'}$ is dominated by the pullback of $\| \ \|_{\Omega, X/S}$ and the two seminorms are equal if for any $s' \in S'$ with $s = g(s')$ the extension $\mathcal{H}(s')/\mathcal{H}(s)$ is universally spectral. In particular, this happens when g is a monomorphism (e.g. embedding of a point) or $S' = S \otimes_k l$ for a tame finite extension l/k.*

Remark 8.1.2. The Kähler seminorm can drop under wildly ramified ground field extensions. Therefore, it makes sense to also introduce the *geometric Kähler seminorm* $\| \ \|_{\overline{\Omega}, X/S}$ obtained by computing the Kähler seminorm after the ground field extension $\widehat{k^a}/k$. By the above theorem the two norms coincides when k has no wildly ramified extensions.

8.2. **Monomiality of the geometric Kähler seminorm.** Assume that $k = k^a$ and X is good. Then for any monomial point $x \in X$ there exists a family of tame parameters at x, i.e. elements $t_1, \dots, t_n \in \mathcal{O}_{X,x}$ such that $n = \dim_x(X)$ and $\mathcal{H}(x)$ is tame over its subfield $k(\widehat{t_1, \dots, t_n})$.

Theorem 8.2.1. *Keep the above notation, then $\frac{dt_1}{t_1}, \dots, \frac{dt_n}{t_n}$ form an orthonormal basis of $\widehat{\Omega}_{\mathcal{H}(x)/k}$.*

The theorem allows to compute the Kähler seminorm on the sheaf $\mathcal{F} = S^n(\Omega^l_{X/k})$ in terms of a tame parameter family: one represents $\phi_x \in \mathcal{F}_x$ as $\sum_e \phi_e e$ with e's of the form $\left(\frac{dt_{i_1}}{t_{i_1}} \wedge \dots \wedge \frac{dt_{i_l}}{t_{i_l}}\right) \otimes \dots \otimes \left(\frac{dt_{j_1}}{t_{j_1}} \wedge \dots \wedge \frac{dt_{j_l}}{t_{j_l}}\right)$ and computes $\|\phi\|_x = \max_e |\phi_e|_x$. Using (non-trivial) results on existence of tame parameters on skeletons one also obtains the following important result.

Corollary 8.2.2. *If $\phi \in S^n(\Omega^l_{X/k})$ is a differential pluriform on X then its geometric Kähler seminorm $\|\phi\|$ restricts to a PL function on any PL subspace of X.*

Remark 8.2.3. Probably, this result also holds for Kähler seminorms, but this requires a new argument.

8.3. **The maximality locus.**

Theorem 8.3.1. *If X possesses a semistable model $\mathfrak{X}$ then for any pluricanonical form $\phi \in \Gamma(\omega_{\mathfrak{X}}^{\otimes m})$ the maximality locus of the geometric Kähler seminorm of ϕ is a union of faces of the skeleton associated with $\mathfrak{X}$.*

Remark 8.3.2. (i) The same result should hold for any log smooth model $\mathfrak{X}$.

(ii) As we know from Example 7.2(ii), the same assertion completely fails for the usual Kähler seminorm.

Corollary 8.3.3. *Assume that $\mathrm{char}(\widetilde{k}) = 0$, X is strictly analytic and rig-smooth and $\phi \in \Gamma(\omega_{\mathfrak{X}}^{\otimes m})$. Then the maximality locus of the Kähler seminorm of ϕ is a PL subspace of X.*

Proof. We can replace k with $\widehat{k^a}$ since this does not affect the Kähler seminorm thanks to the $\mathrm{char}(\widetilde{k}) = 0$ hypothesis. By local uniformization of Berkovich spaces of equal characteristic zero, there exists an admissible covering of X by affinoid domains that possess semistable reduction. It remains to use the above theorem. □

8.4. **Comparison with the weight function.** It remains to answer the natural question whether our definition of Kähler seminorms on pluricanonical sheaves coincides (up to a constant factor) with the definitions of [KS06] and [MN13]. Surprisingly, this is so only in the case of residue characteristic zero. In general, the discrepancy is described by the log-different. For any (not necessarily algebraic) extension l/k of real-valued fields we define the log different $\delta^{\log}_{l/k}$ to be the torsion content of the module $\Omega^{\log}_{l^\circ/k^\circ}$ (in the discrete valued case this is the absolute value of the zeroth Fitting ideal of the torsion part of $\Omega^{\log}_{l^\circ/k^\circ}$). The log different is an important invariant of the extension l/k that measures its “wildness”.

Theorem 8.4.1. *Assume that k is discretely valued with a uniformizer π_k, X is a smooth algebraizable Berkovich space over k, $\phi \in \omega_X^{\otimes m}$, and $x \in X$ is a monomial point; in particular, the weight function wt_ϕ is defined at x. Then $\mathrm{wt}_\phi(x) = |\pi_k|^m (\delta^{\log}_{\mathcal{H}(x)/k})^m \|\phi\|_x$, where the right-hand side involves the Kähler seminorm on $\omega_X^{\otimes m}$.*

In particular, all reasonable seminorms coincide when k has no wildly ramified extensions. (For a discrete valued k this means that $\mathrm{char}(\widetilde{k}) = 0$.) Note also that the weight function drops under wildly ramified extensions, and one can define geometric weight function analogously to geometric Kähler seminorm. Then it follows from the theorem that the geometric Kähler norm coincides with the geometric weight function in the case when the latter is defined.

References

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