SKELETON AND DUAL COMPLEX

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This is a note to the talk I gave in Simons Symposium on Non-Archimedean and Tropical Geometry, held on February 2-6, organized by Matthew Baker and Sam Payne. I want to thank them for the invitation!

Here we give a map on the recent joint works [dFKX12], [NX13] and [KX15], which is our attempt to study the dual complex and establish its relationship with the topology of the analytification of a variety defined over $K=\mathbb{C}[[t]]$. We also ask some questions which are worthy to be further studied.

1. Essential skeleton

Let X be a normal variety, and D a Weil divisor on X. If $D=\sum D_i$ has the following two properties

(1) An irreducible component W of intersections $\bigcap_{j=1,\ldots,k} D_{i_j}$ are all normal.
(2) W is of codimension k in X.

Then we can construct a *dual complex* $\mathcal{D}(D)$ by associate W a k-dimensional cell attaching on the vertices $v_{i_1},\ldots v_{i_k}$.

A very useful category of pairs satisfying the above assumption comes from a dlt pair (X,Δ), where we choose $D=\Delta^{=1}$.

Definition 1.1. A log pair (X,Δ) is called divisorial log terminal (dlt), if there is an open set $U\subset X$ such that U is smooth and $\Delta|_U$ is a reduced simple normal crossing divisor, and for any divisorial valuation E with center on $X\setminus U$, we have $a(E,X,\Delta)>-1$.

Then we know that $D=\Delta^{=1}$ satisfies our assumptions (1) and (2), so we can define $\mathcal{D}(D)$.

Definition 1.2. For any log pair (Y,E), if K_Y+E is $\mathbb{Q}$-Cartier, then we can define a partial resolution, called *dlt modification* $f\colon (X,\Delta)\to Y$ satisfying

(1) Let Δ be the sum of the birational transform of E and the reduced exceptional divisor, then (X,Δ) is dlt.
(2) $K_X+\Delta$ is f-nef.

In [dFKX12], we investigate how the dual complex changes under the minimal model program. As a corollary, we show in various cases, the modification is indeed a collapse (which is a special kind of deformation retract). In particular, we have the following which gives interesting results in the Berkovich space setting.

Theorem 1.3 ([dFKX12])**.** *Let X_K be a smooth projective variety over $K=k((t))$. Let $\mathfrak{X}_i$ (i=1,2) be two projective models over an algebraic curve whose base changes give X_K. Assume there is a morphism $f\colon \mathfrak{X}_1\to\mathfrak{X}_2$ and $(\mathfrak{X}_i,(X_i)_{\rm red})$ are dlt, where X_i are the special fibers.*

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Then $\mathcal{D}(X_1)$ collapses to $\mathcal{D}(X_2)$.

When $(\mathfrak{X}_i, (X_i)_{\rm red})$ are snc, this kind of results can be obtained by weak factorization theorem. But for the general case, we need to invoke the minimal model program.

Now if $\mathfrak{X}$ is a smooth model which is a base change of a relatively projective algebraic model $\mathcal{X}$, such that $(\mathfrak{X}, X_{\rm red})$ is dlt. Assume K_{X_K} is semi ample, i.e., $|mK_{X_K}|$ is base point free for some $m > 0$. Then we can run a relative minimal model program to obtain $\mathfrak{X}^{\rm min}$, which satisfies that $(\mathfrak{X}^{\rm min}, X^{\rm min}_{\rm red})$ is dlt and $K_{\mathfrak{X}^{\rm min}} + X^{\rm min}_{\rm red}$ is relatively semi ample. This model $\mathfrak{X}^{\rm min}$ is important, since we have the following

Theorem 1.4 ([NX13]). *The Kontsevich-Soibelman essential skeleton of $X_K^{\rm an}$ is naturally isomorphic to $\mathcal{D}(X^{\rm min}_{\rm red})$.*

Recall that the Kontsevich-Soibelman essential skeleton defined in [MN12] is a natural subspace embedded in $X^{\rm an}$, which does not depend on the choice of the models.

Corollary 1.5. *The analytic space $X_K^{\rm an}$ admits a deformation retract to the essential skeleton.*

Here the deformation retract we choose depends on the MMP process, which is in general not unique. So it is natural to ask

Question 1.6. *Does there exist a more canonically defined way to yield the deformation retract?*

2. Topology of the dual complex

In [dFKX12], we show the following

Theorem 2.1. *If X_K is a rationally connected variety, then $\mathcal{D}(X_{\rm red})$ is always contractible.*

Later this result is also used in a relative setting in [BF14]. One interesting question is that the process of minimal model program indeeds yields a special component, called the *Kollár component*. It depends on the MMP process, so in general it is not unique. However, it has been shown that it carries interesting geometric properties. A natural question is

Question 2.2. *Study Kollár component from the Berkovich viewpoint.*

When X_K is of general type, we do not know many non-trivial restrictions on $\mathcal{D}(X_{\rm red})$. So the most interesting case seems to be on the border line when X_K is a Calabi-Yau, which also natural appears in many other questions.

A probably naive question is the following,

Question 2.3. *If X_K is a simply connected Calabi-Yau manifold, such that*

$$H^i(X_K, \mathcal{O}) = 0 \text{ for any } 0 < i < \dim X.$$

Let $\mathfrak{X}$ be a semistable model with maximal degeneration and $K_{\mathfrak{X}} \sim 0$, then $\mathcal{D}(X)$ is isomorphic to the sphere $S^{\dim X}$.

This question is far from known to be true. In a more general setting, given any K_{X_K} with $K_{X_K} \sim_{\mathbb{Q}} 0$, we can study the dual complex of $\mathcal{D}(X^{\rm min}_{\rm red})$. For any $E \subset X^{\rm min}_{\rm red}$, the link of each vertex v_E in this complex is given by $\mathcal{D}(D_E)$. Here we define

$$(K_{\mathfrak{X}^{\rm min}} + X^{\rm min}_{\rm red})|_E = K_E + \Delta_E,$$

and $D_E = \Delta_E^{=1}$. Thus (E, Δ_E) is a dlt log Calabi-Yau pair.

Then it is natural to study $\mathcal{D}(D)$ for any dlt log Calabi-Yau pair (X, Δ). Using MMP, we obtain the following results

Theorem 2.4 ([KX15]). *Let (X, Δ) be a dlt log Calabi-Yau pair and $D = \Delta^{=1}$. Then*

(1) *$H^i(\mathcal{D}(D), \mathbb{Q}) = 0$ for $0 < i < \dim \mathcal{D}(D)$.*
(2) *If $\dim \mathcal{D}(D) > 1$, then $\pi_1(X^{\rm sm})$ admits a surjection to $\pi_1(\mathcal{D}(D))$. In particular, the pro-finite completion $\hat{\pi}_1(\mathcal{D}(D))$ is finite.*

Corollary 2.5. *Question 2.3 has an affirmative answer when* $\dim(X_K) \leq 4$.

The main idea of proving Theorem 2.4 is using MMP to construct a new birational log Calabi-Yau lc model (X', Δ'), such that D' supports an ample divisor. Then after doing some standard birational modification, we can conclude.

Since after Theorem 2.4, we have a good understanding of the rational homology and fundamental group, the remaining important question is

Question 2.6. How to compute $H^i(\mathcal{D}(D), \mathbb{Z})$? Or similarly, how to compute $H^i(\mathcal{D}(X^{\rm min}_{\rm red}), \mathbb{Z})$? Is $H^i(D(X), \mathbb{Z}) = 0$ in the setting of Question 2.3 for $0 < i < \dim(X_K)$?

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