Problem 1.1. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \geq 0}, \mathbb{P})\) be a filtered probability space and \(T\) a stopping time.

1. Show that
   \[ \mathcal{F}_T := \{ A \in \mathcal{F} | A \cap \{ T = n \} \in \mathcal{F}_n \ (\forall \ n) \} \]
   is a sigma-field.

2. If \(\{X_n\}_n\) is an adapted process, then the process \(\{X_{n \wedge T}\}_n\) is also adapted, and the random variable \(X_T 1_{\{T < \infty\}}\) is \(\mathcal{F}_T\)-measurable.

Solution:
(1) this item is just an easy application of the definition of sigma-algebra.
(2) Let \(B\) any Borel measurable set of \(\mathbb{R}\) (or, in general, any measurable set of the state space). The we have
   \[ \{X_{n \wedge T} \in B\} = \bigcup_{k=0}^{n-1} \left( \{T = k\} \cap \{X_k \in B\} \right) \cup \left( \{T \geq n\} \cap \{X_n \in B\} \right) \in \mathcal{F}_n. \]
In addition
   \[ \{X_T 1_{\{T < \infty\}} \in B\} \cap \{T = n\} = \{X_n \in B\} \cap \{T = n\} \in \mathcal{F}_n \]
for each \(n\), so
   \[ \{X_T 1_{\{T < \infty\}} \in B\} \in \mathcal{F}_T \]
by the definition of \(\mathcal{F}_T\).

Problem 1.2. Let \(0 \leq S \leq T\) two stopping times with respect to the discrete-time filtration \((\mathcal{F}_n)_{n \geq 0}\). Let \(A \in \mathcal{F}_S\). Show that the random time \(T'\) defined by
   \[ T' = S 1_A + T 1_{A^c} \]
is also a stopping time.

Solution: We have that
   \[ \{T' = n\} = \left( \{S = n\} \cap A \right) \cup \left( \{S \leq n\} \cap \{T = n\} \cap A^c \right) \in \mathcal{F}_n \]
since
   \[ \{S \leq n\} \cap A^c \in \mathcal{F}_n. \]

Problem 1.3. (1) (Doob’s decomposition). Let \(\{X_n, \mathcal{F}_n\}\) be a submartingale. Show that it can be decomposed uniquely as
   \[ X_n = M_n + A_n, \ n = 0, 1, \ldots \]
where \(\{M_n, \mathcal{F}_n\}\) is a martingale and the process \(\{A_n\}_n\) is increasing, predictable with respect to the filtration \(\{\mathcal{F}_n\}_n\) and \(A_0 = 0\). Find the processes \(M\) and \(A\) in terms of the process \(X\).

2. Let \(X_n = \sum_{k=1}^{n} I_{B_k}\) for \(B_n \in \mathcal{F}_n\). What is the Doob decomposition for \(X_n\)?

Solution: 1. Use the decomposition relation for \(n + 1\) and \(n\) and subtract term by term to get
   \[ X_{n+1} - X_n = M_{n+1} - M_n + (A_{n+1} - A_n) \]
Taking conditional expectation with respect to \(\mathcal{F}_n\) and using the martingale property for \(M\) and the fact that \(A\) is predictable we obtain
   \[ \mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = A_{n+1} - A_n. \]
Now
   \[ A_n = A_0 + (A_1 - A_0) + \cdots + (A_n - A_{n-1}) = \mathbb{E}[X_1 - X_0 | \mathcal{F}_0] + \cdots + \mathbb{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}]. \]
We can also see that

\[ M_{n+1} - M_n = X_{n+1} - X_n - (A_{n+1} - A_n) = X_{n+1} - \mathbb{E}[X_{n+1} | \mathcal{F}_n] \]

and since \( M_0 = X_0 \) we have

\[ M_n = X_0 + (X_0 - \mathbb{E}[X_1 | \mathcal{F}_0]) + \cdots + (X_n - \mathbb{E}[X_n | \mathcal{F}_{n-1}]). \]

So far we have only showed uniqueness, but existence of this decomposition can be easily proven taking these relations as definitions for \( M \) and \( A \).

2. Since \( X_n - X_{n-1} = 1_{B_n} \) and \( \mathbb{E}[1_{B_n} | \mathcal{F}_{n-1}] = p(B_n | \mathcal{F}_{n-1}) \), we conclude that

\[ A_n = \sum_{i=1}^{n} p(B_i | \mathcal{F}_{i-1}) \]

and

\[ M_n = X_n - A_n = \sum_{i=1}^{n} (1_{B_i} - p(B_i | \mathcal{F}_{i-1})) \]

**Problem 1.4.** Let \( Y_1, Y_2, \ldots \) be non-negative i.i.d. with \( \mathbb{E}[Y_n] = 1 \) and \( \mathbb{P}(Y_n = 1) < 1 \).

1. Show that \( X_n = \prod_{k=1}^{n} Y_k \) is a martingale.
2. Use the martingale convergence theorem to conclude that \( X_n \to 0 \) a.s.
3. Use the SLLN to show that

\[ \frac{\log(X_n)}{n} \to c < 0, \text{ a.s.} \]

**Solution:**

1. \( \mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_n Y_{n+1} | \mathcal{F}_n] = X_n \mathbb{E}[Y_{n+1} | \mathcal{F}_n] = X_n \), since \( Y_{n+1} \) is independent on the sigma algebra \( \mathcal{F}_n \) generated by \( X_1, \ldots, X_n \).

2. (direct proof) One can prove this part directly, using martingale convergence theorem for \( X_n \). First, we conclude that \( X_n \to X_\infty \geq 0 \). Now, on the event \( \{X_\infty > 0\} \) we have that \( Y_n = X_n / X_{n-1} \to 1 \). The hypothesis ensures that \( \mathbb{P}(Y \geq 1 + \varepsilon) > \varepsilon \) for some small \( \varepsilon > 0 \). Borel-Cantelli implies that \( \mathbb{P}(Y_n \geq 1 + \varepsilon \text{ i.o}) = 1 \), so \( \mathbb{P}(Y_n \to 1) = 0 \). Therefore, \( \mathbb{P}(X_\infty > 0) = 0 \) as well.

3. We have that \( \log(X_n) = \log(Y_1) + \cdots + \log(Y_n) \). The hypotheses ensures that \( (\log Y)^+ \in L^1 \) (by Jensen) and \( \mathbb{E}[\log Y] < 0 \) (it CAN be \( -\infty \)). If the expectation is finite, then we can apply the SLLN. If the expectation is \( -\infty \), we can TRUNCATE \( \log Y \) at a negative level \( -C \) and then let \( C \to \infty \) to conclude that

\[ \frac{\log(Y_1) + \cdots + \log(Y_n)}{n} \to \mathbb{E}[\log Y] \in [-\infty, 0). \]

2. (alternative proof, based on 3.) Since \( \log(X_n)/n \to c < 0 \) then \( X_n \to 0 \).

**Problem 1.5.** (An example of a Uniformly Integrable Martingale which is not in \( \mathcal{H}^1 \)). Consider \( \Omega = \mathbb{N}, \mathcal{F} = 2^\mathbb{N} \) and the probability measure \( \mathbb{P} \) defined by

\[ \mathbb{P}[k] = 2^{-k}, \quad k = 1, 2, \ldots. \]

Consider now the random variable \( K \) such that \( K(k) = k, k = 1, 2, \ldots. \)

1. find an explicit example of a random variable \( Y : \Omega \to [1, \infty) \) such that \( \mathbb{E}[Y] < \infty \) and \( \mathbb{E}[Y K] = \infty. \)
2. consider the filtration

\[ \mathcal{F}_n = \sigma\{\{1, \{2, \ldots, \{n-1, \{n, n+1, \ldots\}\}\}\}\} \]

Find an expression for \( X_n = \mathbb{E}[Y | \mathcal{F}_n] \).

3. Show that the martingale \( (X_n)_n \) is not in \( \mathcal{H}^1 \) (note that the martingale is UI by the way it is defined).

**Solution:**
Taking into account that

\[ X_n(k) = Y(k), \quad k = 1, 2, \ldots, n - 1 \]

and

\[ X_n(n) = X_n(n + 1) = \cdots = 2^{-1}Y(n) + 2^{-2}Y(n + 1) + \ldots \]

we see that \( X^*(n) \geq X_n(n) \), so

\[ \mathbb{E}[X^*(n)] \geq \sum_{n=1}^{\infty} 2^{-n} \left( \sum_{k=n}^{\infty} 2^{-k-1} 2^k \frac{1}{k^2} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{1}{k^2} = \infty. \]

**Problem 1.6.**

1. Suppose \( (X_n)_n \) is a submartingale such that \( \mathbb{E}[|X_{n+1} - X_n|_F] \leq B \) a.s for each \( n \) where \( B \) is a constant. Show that if \( N \) is a stopping time such that \( \mathbb{E}[N] < \infty \) then \( (X_n \wedge N)_n \) is u.i., so, according to the optional sampling theorem \( \mathbb{E}[X_0] \leq \mathbb{E}[X_N] \).
2. (Wald I): Let \( \xi_1, \xi_2, \ldots \) be i.i.d such that \( \xi_i \in L^1 \). If \( S_n = \xi_1 + \ldots + \xi_n \) and \( N \) is a stopping time such that \( \mathbb{E}[N] < \infty \) then

\[ \mathbb{E}[S_N] = \mathbb{E}[N]\mathbb{E}[\xi] \]

Please note that this implies that the first hitting time at level one for the symmetric random walk has INFINITE expectation.

**Solution:** 1. Suppose \( |X_{n \wedge N}| \leq |X_0| + |X_1 - X_0| + \ldots + |X_n - X_{n-1}| = X_0 + \sum_{n=0}^{\infty} |X_{n+1} - X_n|_{\{N>n\}} \). Taking into account that \( \{N>n\} \in \mathcal{F}_n \), we can condition each term on time \( n \) to conclude that

\[ \mathbb{E}[\sup_{n} |X_{n \wedge N}|] \leq \mathbb{E}[X_0] + \sum_{n=0}^{\infty} \mathbb{E}[|X_{n+1} - X_n|_{\mathcal{F}_n}]_{\{N>n\}} \leq \mathbb{E}[X_0] + B \sum_{n=0}^{\infty} \mathbb{P}(N>n) = \mathbb{E}[X_0] + \mathbb{P}(N=0) + \mathbb{E}[N] < \infty. \]

This means that the sequence \( (X_{n \wedge N})_n \) is dominated by an

2. Define the process \( M_n = S_n - n\mathbb{E}[\xi] \). We have \( M_{n+1} - M_n = \xi_{n+1} - \mathbb{E}[\xi_{n+1}] \), so \( \mathbb{E}[M_{n+1} - M_n_{\mathcal{F}_n}] = 0 \), and

\[ \mathbb{E}[(M_{n+1} - M_n)_{\mathcal{F}_n}] \leq \mathbb{E}[|\xi - \mathbb{E}[\xi]|] \]

This means that \( M \) is a martingale and we can apply Part 1 to get \( \mathbb{E}[M_N] = 0 \) which means \( \mathbb{E}[S_N] = \mathbb{E}[N]\mathbb{E}[\xi] \).

**Problem 1.7.** (Quadratic variation of square integrable martingales). Consider a square integrable martingale, by which we mean that \( X_0 = 0 \) and \( X_n \in L^2 \) for each \( n \). We can define the square bracket (or the quadratic variation) of the martingale by

\[ [X, X]_n = \sum_{k=1}^{n} (X_k - X_{k-1})^2. \]

At the same time, we define the angle bracket, or the predictable quadratic variation process \( \langle X, X \rangle \) as the predictable increasing process in the Doob decomposition

\[ X^2 = M + \langle X, X \rangle. \]

Show that

1. \( \langle X, X \rangle_n = \sum_{k=1}^{n} \mathbb{E}[(X_k - X_{k-1})^2_{\mathcal{F}_{k-1}}] \) and the predictable process \( \langle X, X \rangle \) is the compensator in the Doob decomposition of \( [X, X] \).
(2) let $T$ be a stopping time (possibly unbounded and infinite). Show that
\[ \mathbb{E}[(X,X)_T] = \mathbb{E}[(X,X)_n]. \]
(note that the random variables above are actually well defined)

(3) show that
\[ \mathbb{E} [ \sup_{0 \leq k \leq T} |X_k|^2 ] \leq 4 \mathbb{E}[(X,X)_T] = 4 \mathbb{E}[(X,X)_n]. \]

(4) assume that
\[ \mathbb{E}[(X,X)_T] < \infty. \]
Then the processes
\[ X^2_{n \wedge T} - [X,X]_{n \wedge T}, \ n = 0, 1, 2 \ldots, \]
\[ X^2_{n \wedge T} - (X,X)_{n \wedge T}, \ n = 0, 1, 2 \ldots, \]
are uniformly integrable martingales. In particular, this means that $X_\infty$ is well defined on the set $\{ T = \infty \}$ for such a stopping time $T$ and
\[ \mathbb{E}[X^2_T] = \mathbb{E}[(X,X)_T] = \mathbb{E}[(X,X)_\infty]. \]

(5) show that $X_\infty = \lim_n X_n$ exists on and is finite on $\{ (X,X)_\infty < \infty \}.$

Solution:

(1) Using the "fundamental property of martingales", we can easily check that $X^2 - [X,X]$ is a martingale. Therefore, the predictable compensator of $X^2$ (in the Doob decomposition) is the same as the predictable compensator of $[X,X]$. Writing now the explicit representation for the predictable compensator of $[X,X]$ we get the answer.

(2) according to the item above, $[X,X] - \langle X,X \rangle$ is a martingale. In particular, for each fixed $n$ we have
\[ \mathbb{E}[(X,X)_{n \wedge T}] = \mathbb{E}[(X,X)_{n \wedge T}]. \]
We can now let $n \to \infty$ and use Monotone Convergence.

(3) We apply the Maximal Inequality for the stopped process $X^n_T$ up to time $n$, and use the martingale property of the process $X^2 - [X,X]$ stopped at $T$ between time $0$ and $n$. After that we let $n$ go to infinity.

(4) using the previous item, we see that both random variables in the difference are bounded by something integrable (as in class for continuous martingales)

(5) let $T_k = \inf \{ n | (X,X)^n_{n+1} \geq k \}$. Because $(X,X)$ is predictable, then $T_k$ is a stopping time. We also have that
\[ \langle X,X \rangle_{T_k} \leq k, \ \{ (X,X)_\infty < k \} \subset \{ T_k = \infty \}. \]
According to part 4, $X_\infty = \lim_n X_n$ is well defined on $\{ T_k = \infty \}$, so the conclusion follows by letting $k \to \infty$.

Problem 1.8. (Wald II) Use the notation of exercise 9.4.2 and assume that $\mathbb{E}[\xi] = 0$ and $\mathbb{E}[\xi^2] = \sigma^2 < \infty$. Use exercise 9.5.3 to show that if $N$ is a stopping time such that $\mathbb{E}[N] < \infty$ then $\mathbb{E}[S^2_N] = \sigma^2 \mathbb{E}[N]$.

Solution: Since $\mathbb{E}[\xi] = 0$ it is an easy exercise to show that $(S_n)_n$ is a martingale. Denote $Y_n = S_n - n \sigma^2$. We have that
\[ Y_{n+1} - Y_n = S^2_{n+1} - S^2_n - \sigma^2 = (S_n + \xi_{n+1})^2 - S^2_n - \sigma^2 = 2S_n \xi_{n+1} + \xi^2_{n+1} - \sigma^2, \]
so
\[ \mathbb{E}[Y_{n+1} - Y_n | F_n] = 2S_n \mathbb{E}[\xi_{n+1}] + \mathbb{E}[\xi^2_{n+1}] - \sigma^2 = 0, \]
which means that $Y$ is a martingale. This shows that the process $A_n = n \sigma^2$ is the predictable quadratic variation of the martingale $S$. We can apply now the previous problem to finish the proof.
Problem 1.9. (de la Vallée Poussin criterion) A set of random variables \((X_i)_{i \in I}\) is u.i. if and only if there exists a function \(\psi : [0, \infty) \to [0, \infty)\) such that \(\lim_{x \to \infty} \psi(x)/x = \infty\) and 

\[
\sup_{i \in I} \mathbb{E}[^{\psi}(|X_i|)] < \infty.
\]

The function \(\psi\) can be chosen to be convex.

**Solution:** One implication is quite easy. More precisely, if there exists such a function \(\psi\), then, for each \(\epsilon\) there exists \(M(\epsilon)\) such that

\[
\frac{\psi(x)}{x} \geq \frac{1}{\epsilon}, \quad x \geq M(\epsilon).
\]

Now, for any \(M \geq M(\epsilon)\) we have

\[
\mathbb{E}[|X_i|1_{\{|X_i| \geq M\}}] \leq \epsilon \mathbb{E}[\psi(|X_i|)1_{\{|X_i| \geq M\}}] \leq \epsilon \sup_{i \in I} \mathbb{E}[\psi(|X_i|)].
\]

For the other implication, let us recall that, if \(\psi\) is (piecewise) \(C^1\) and \(\psi(0) = 0\) then

\[
\mathbb{E}[\psi(|X|)] = \int_0^\infty \psi'(x)\mathbb{P}(|X| \geq x)dx.
\]

We also have that

\[
\mathbb{E}[|X|1_{\{|X| \geq M\}}] = \int_0^\infty \mathbb{P}(|X|1_{\{|X| \geq M\}} \geq x)dx = \int_M^\infty \mathbb{P}(|X| \geq x)dx.
\]

Using the definition of uniform integrability, we can therefore choose an increasing sequences \(a_n \uparrow \infty\), (with \(a_n\) increasing very slowly), such that

\[
\sum_n a_n \sup_i \int_n^\infty \mathbb{P}(|X_i| \geq a_n)dx < \infty.
\]

We can now define the piece-wise \(C^1\) function \(\psi\) by \(\psi(0) = 0\) and

\[
\psi(x) = a_n, \quad x \in (n, n+1),
\]

and finish the proof.

Problem 1.10. (backward martingales) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\{\mathcal{F}_n\}_{n \geq 0}\) a “decreasing filtration”, which means that \(\mathcal{F}_{n+1} \subset \mathcal{F}_n\). Let \(\{X_n, \mathcal{F}_n\}_n\) be a backward martingale, which means that \(X_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P}), n \geq 0\) and \(X_{n+1} = \mathbb{E}[X_n | \mathcal{F}_{n+1}]\). Show that \(X_n\) converges a.s. and in \(L^1\) to some variable \(X_{\infty}\) and \(X_{\infty} = \mathbb{E}[X_n | \mathcal{F}_{\infty}]\), where \(\mathcal{F}_{\infty} = \cap_n \mathcal{F}_n\).

**Solution:** Using the uncrossing inequality (applied backwards), we know that

\[
X_n \to X_{\infty} \in \mathcal{F}_{\infty}, \text{ a.s.}
\]

Since \(X_n = \mathbb{E}[X_0 | \mathcal{F}_n]\), we know (for example from Hw 1) that \(X_n\) is a UI sequence. This implies that \(X_{\infty}\) is integrable, and \(X_n \to X_{\infty}\) in \(L^1\).

Now, let \(A \in \mathcal{F}_{\infty}\). Let also \(k \geq n\). Since \(X_k = \mathbb{E}[X_n | \mathcal{F}_k]\), we have

\[
\mathbb{E}[X_n 1_A] = \mathbb{E}[X_k 1_A].
\]

Letting \(k \to \infty\) we obtain

\[
\mathbb{E}[X_n 1_A] = \mathbb{E}[X_{\infty} 1_A], \quad (\forall) A \in \mathcal{F}_{\infty},
\]

or

\[
X_{\infty} = \mathbb{E}[X_n | \mathcal{F}_{\infty}].
\]

Problem 1.11. (backward submartingales) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\{\mathcal{F}_n\}_{n \geq 0}\) a “decreasing filtration” as above. Let \(\{X_n, \mathcal{F}_n\}_n\) be a backward submartingale, which means that \(X_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P}), n \geq 0\) and \(X_{n+1} \leq \mathbb{E}[X_n | \mathcal{F}_{n+1}]\). Show that \(\{X_n\}_n\) is u.i. if and only if \(\inf_n \mathbb{E}[X_n] > -\infty\).
**Solution:** We can use a (backwards) upcrossing inequality to conclude that there exists an $X_\infty \in L^1(\mathcal{F}_\infty)$ such that

$$X_n \to X_\infty \text{ a.s.}$$

The positive part $X_n^+$ is UI, so (splitting each $X_n$ into $X_n^+ - X_n^-$) it is enough to show that

$$\mathbb{E}[X_n] \downarrow \mathbb{E}[X_\infty].$$

Actually, just using Fatou for the negative part (since the positive part converges in $L^1$) we have that $\mathbb{E}[X_\infty] \geq \lim_n \mathbb{E}[X_n]$. Now, for $k \leq n$ we have $X_k \leq \mathbb{E}[X_n|\mathcal{F}_k]$. Letting $k \to \infty$ and using the previous problem, we have that

$$X_\infty \leq \mathbb{E}[X_n|\mathcal{F}_\infty],$$

so $\mathbb{E}[X_\infty] \leq \mathbb{E}[X_n]$, finishing the proof.