Problem 4.1. Let $M$ and $N$ be two continuous local martingales on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$. Show that if $M$ and $N$ are independent (as processes), then $\langle M, N \rangle = 0$.

Solution: By stopping at the times

$$T_n = \inf\{0 \leq t < \infty \mid |M_t| \geq n\}$$

and

$$S_n = \inf\{0 \leq t < \infty \mid |N_t| \geq n\}$$

we can assume that both $M$ and $N$ are actually bounded. Please note that it may not be possible to find the SAME localizing sequence for both $M$ and $N$ to still keep them independent. However, this is not an issue.

From here, there are basically two ways to go ahead:

1. show that the discretized cross variation satisfies

$$E[T_{M,N}^\Delta] \rightarrow 0, \text{ as } \|\Delta\| \rightarrow 0.$$

2. alternatively, we can show that the product $MN$ is a martingale. It is, actually, enough to show that the product $MN$ is a martingale with respect to the (possibly smaller) natural filtration generated by both process $M$ and $N$ that we denote

$$(\mathcal{F}^{M,N}_t)_{0 \leq t < \infty}.$$ 

The reason why this is enough is the simple fact that $M$ and $N$ are (still) martingales under the smaller filtration, and the quadratic variation should not depend on the filtration (as long as the filtrations preserves the semi-martingale property).

To sum up, we need to show that, for $s < t$ we have

$$E[M_tN_t - M_sN_s | \mathcal{F}^{M,N}_s] = E[(M_t - M_s)(N_t - N_s) | \mathcal{F}^{M,N}_s] = 0.$$

The first equality above is "the fundamental property of martingales", so we need only prove the second equality. The second equality, can be rewritten by integrating against "test sets" as

$$E[(M_t - M_s)(N_t - N_s) 1_A] = 0, \quad \forall \ A \in \mathcal{F}^{M,N}_s.$$

We make the observation that the sigma-algebra $\mathcal{F}^{M,N}_s$ is generated by the $\pi$-system

$$\{A = B \cap C \mid B \in \mathcal{F}^M_s, C \in \mathcal{F}^N_s\}.$$

Therefore, it is enough to show that

$$E[(M_t - M_s)(N_t - N_s) 1_B 1_C] = 0, \quad B \in \mathcal{F}^M_s, C \in \mathcal{F}^N_s.$$

Now, we can see that $(M_t - M_s)1_B$ and $(N_t - N_s)1_C$ are contingent on the paths of $M$ and $N$, respectively. In other words,

$$(M_t - M_s)1_B \in \mathcal{F}^M_\infty, \quad (N_t - N_s)1_C \in \mathcal{F}^N_\infty,$$

so they are independent. Therefore

$$E[(M_t - M_s)(N_t - N_s) 1_B 1_C] = E[(M_t - M_s)1_B(N_t - N_s)1_C]$$

$$= E[(M_t - M_s)1_B] \times E[(N_t - N_s)1_C].$$

Each of the terms above is actually zero, finishing the proof. For example, by first conditioning on $\mathcal{F}^M_s$, we have

$$E[(M_t - M_s)1_B] = E[1_B E[M_t - M_s | \mathcal{F}^M_s]],$$

but the martingale property says that $E[M_t - M_s | \mathcal{F}^M_s] = 0$. 

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**Problem 4.2.** (Ex 3.14 page 153 in RY) Let $M$ be a continuous adapted process and $A$ a continuous adapted process of finite variation with $A_0 = 0$. If, for every $\lambda$, the process

$$Z_t = e^{\lambda M_t - \frac{\lambda^2}{2} A_t},$$

is a local martingale, then $M$ is a local martingale and $(M, M) = A$.

**Solution:** By stopping, we can assume that both $M$ and $A$ are bounded. i.e. $|M|, |A| \leq K$ for some $K$. In this case, $Z$ is of class (DL) and, therefore, a martingale (for each $\lambda$). For $0 \leq s \leq t$, we have

$$\mathbb{E} \left[ e^{\lambda M_t - \frac{\lambda^2}{2} A_t} | \mathcal{F}_s \right] = e^{\lambda M_s - \frac{\lambda^2}{2} A_s}.$$  

Formally, we can take the derivative of the above relation with respect to $\lambda$, at $\lambda = 0$. Rigorously, we have

$$\mathbb{E} \left[ e^{\lambda M_t - \frac{\lambda^2}{2} A_t} \frac{1 - \lambda}{\lambda} | \mathcal{F}_s \right] = \frac{e^{\lambda M_s - \frac{\lambda^2}{2} A_s} - 1}{\lambda}.$$  

Now, $D_\lambda \to M_t$ a.s. and, using mean value Theorem, $D_\lambda$ is uniformly bounded. We can apply the Dominated Convergence (conditional version) for the LHS to conclude that $M$ is a martingale. In order to show that $A = \langle M \rangle$, we apply Itô formulate to $Z$, for $\lambda = 1$. We have

$$dZ_t = Z_t (dM_t - \frac{1}{2} dA_t + \frac{1}{2} d\langle M \rangle_t).$$

Since $M$ is a local martingale, and the other parts are continuous with bounded variation, the only way for $Z$ to be a local martingale is for $A = \langle M \rangle$.

**Problem 4.3.** If $X$ and $Y$ are two continuous semimartingales starting at 0, relate the processes $\mathcal{E}(X)\mathcal{E}(Y)$ and $\mathcal{E}(X + Y)$. When are they equal?

**Solution:**

$$\mathcal{E}(X + Y) = e^{X+Y-\frac{1}{2}(X+Y,X+Y)} = e^{X-\frac{1}{2}(X)} e^{Y-\frac{1}{2}(Y)} e^{-\langle X, Y \rangle} = \mathcal{E}(X)\mathcal{E}(Y) e^{-\langle X, Y \rangle}.$$  

Obviously, the equality holds only if $\langle X, Y \rangle = 0$.

**Problem 4.4.** (Ex 3.25, page 157 RY) If $M$ is a continuous local martingale such that $M_0 = 0$ then

$$\{\mathcal{E}(M)_\infty = 0\} = \{\langle M, M \rangle_\infty = \infty\}$$  

(Hint : $\mathcal{E}(M) = \mathcal{E}(\frac{1}{2}M)^2 e^{-\frac{1}{4}(M,M)}$)

**Solution:** First note that the equality in the hint follows easily from the above problem applied to $X = Y = \frac{1}{2} M$. Next, $\mathcal{E}(M)$ is a positive super-martingale, so it has an a.s. FINITE limit at infinity, that we denote by $\mathcal{E}(M)_\infty$. The same is true for $\mathcal{E}(\frac{1}{2} M)$, so the above relation clearly shows that $\langle M \rangle_\infty = \infty$ implies that $\mathcal{E}(M)_\infty = 0$.

Recall that, on $\{\langle M \rangle_\infty < \infty\}$ we have that the local martingale has a finite limit $M_\infty$. Therefore, on this event,

$$\mathcal{E}(M)_\infty = e^{M_\infty - \frac{1}{2}(M)_\infty} > 0.$$  

In other words, $\langle M \rangle_\infty < \infty$ implies that $\mathcal{E}(M)_\infty > 0$, ending the proof.

**Problem 4.5.** Exercise 4: (Linear Equations) If $Y$ and $H$ are two CONTINUOS semimartingales, solve in closed form the linear equation for $X$

$$\begin{cases}
    dX_t = X_t dY_t + dH_t \\
    X_0 = \xi
\end{cases}$$
**Solution:** Choose the integrating factor $Z = e^{-Y+\frac{1}{2}(Y)} = \mathcal{E}(Y)^{-1}$. The process $Z$ satisfies

$$dZ_t = Z_t(-dY_t + (dY_t)^2).$$

Now, we apply Itô to $XZ$ and simplify to obtain

$$d(X_tZ_t) = X_t dZ_t + Z_t dX_t + dX_t dZ_t = Z_t dH_t - Z_t d\langle Y,H \rangle_t.$$ 

Therefore,

$$X_t Z_t = \xi + \int_0^t Z_s dH_s - \int_0^t Z_s d\langle Y,H \rangle_s.$$ 

Therefore,

$$X_t = Z_t^{-1} \left( \xi + \int_0^t Z_s dH_s - \int_0^t Z_s d\langle Y,H \rangle_s \right) = \mathcal{E}(Y)_t \left( \xi + \int_0^t \mathcal{E}(Y)_s^{-1} dH_s - \int_0^t \mathcal{E}(Y)_s^{-1} d\langle Y,H \rangle_s \right).$$