Problem 6.2. Let $Z$ be an exponential martingale. For example, $dZ_t = \exp(\mu W_t - \frac{1}{2} \mu^2 t)$ and the the measure $P^\mu$ on $\mathcal{F}_\infty$ with density process $Z$ (recall that $P^\mu$ is not absolutely continuous with respect to the original measure $P$). If $T$ is a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, finite a.s (which means $P[T < \infty] = 1$), show that 
\[ E[\exp(\mu W_T - \frac{1}{2} \mu^2 T)] = 1 \]
if and only if 
\[ P^\mu[T < \infty] = 1. \]
Consider the more general case: on a space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{0 \leq t < \infty}$ the measure $Q$ is a probability measure on $\mathcal{F}_\infty$ which has density process $Z$ with respect to the original measure $P$, i.e. \( \frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t \) for each $t < \infty$. If $T$ is a stopping time such that $P[T < \infty] = 1$, do we still have 
\[ E[Z_T] = 1 \] if and only if $Q[T < \infty] = 1$?

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Problem 6.1. (Different notions of orthogonality for Martingales) Consider a (finite or even infinite) time horizon $T$, and a probability space and a filtration satisfying usual conditions. Two martingales $M, N \in \mathcal{H}_2^2$ (square integrable, starting at position zero) are called

- $L^2$-orthogonal if $E[M_T N_T] = 0$ (definition in class)
- strongly orthogonal if $(M_t N_t)_{0 \leq t \leq T}$ is a martingale (in class as well) and
- weakly orthogonal if $E[M_t N_t] = 0$ for each $0 \leq t \leq T$ (additional definition).

(1) show that the following are equivalent
(a) $M, N$ are weakly orthogonal
(b) $E[M_s N_t] = 0$ for each $0 \leq s, t \leq T$
(c) $E[M_s N_t] = 0$ for each $0 \leq s \leq T$ and each stopping time $s \leq \rho \leq T$.

(2) show that the following are equivalent
(a) $M, N$ are (strongly) orthogonal
(b) for each stopping times $0 \leq \rho, \tau \leq T$ we have $E[M_\rho N_\tau] = 0$
(c) for each stopping times $0 \leq \rho, \tau \leq T$ the stopped martingales $M^\rho$ and $N^\tau$ are strongly orthogonal
(d) $E[M_s N_\tau] = 0$ for each $0 \leq s \leq T$ and each stopping time $\tau \leq s$.

(3) give examples of martingales which are weakly orthogonal but not orthogonal (hint: consider integrals with respect to a Brownian Motion).

Solution: Part 1 and 2 are based on rather simple manipulations related to conditioning, the tower property and the definition of martingales.

For part 3, one should note that, IN THE CASE $M, N$ are continuous, the weak orthogonal property mean that $E[(M, N)_t] = 0$ for all $t$, while the strong orthogonal property means $(M, N) = 0$. This suggest how to construct an example. We only to this for fixed time horizon $T < \infty$ (but it is obvious how to extend this to $T = \infty$.) Choose $M = W$ (some Brownian motion), and $N_t = \int_0^t H_s dW_s$ for any process $H$ such that

\[ E[\int_0^T H_s^2 ds] < \infty, \quad E[H_s] = 0 \quad \forall s \in [0, T]. \]

For example, $H_s = sgn(W_s)$.
Solution: The answer is yes in the more general case. Assume that we have a measure $Q$ who may NOT be absolutely continuous with respect to $P$ at time $t = \infty$, but is absolutely continuous at any finite time $t$. For each $t < \infty$ we now have

$$Q(T < t) = \mathbb{E}[1_{\{T < t\}}Z_t] = \mathbb{E}[1_{\{T < t\}}Z_T].$$

Letting $t \to \infty$ at taking into account that $P(T < \infty) = 1$ we get

$$Q(T < \infty) = \mathbb{E}[Z_T].$$

Problem 6.3. With the notations from the exercise above, if $T_b$ is the first time the Brownian Motion $W$ hits level $b$, show that

$$\mathbb{E}[\lambda - \alpha T_b] = e^{\mu b} - \lambda, \alpha > 0.$$  

Solution: Please note that, unlike in class, here we start with a BM and its hitting time and then we change the measure. The meaning is, obviously, the same.

Under $P$, $W$ is a Brownian Motion with drift $\mu$, since $W_t = \mu t + (W_t - \mu t)$.

Assume $b > 0$

Fix $\lambda > 0$. We have, under $P$ the exponential martingale

$$e^{\lambda W_t - (\lambda + \frac{\lambda^2}{2})t}.$$

Using now the Optional Sampling Theorem, we have

$$1 = \mathbb{E}^{\mu}[e^{\lambda W_t - (\lambda + \frac{\lambda^2}{2})t}].$$

If, in addition, $\alpha := \lambda + \frac{\lambda^2}{2} > 0$, then since the exponential process above is bounded before $T_b$ we obtain, as we have done in class for the BM (without drift) that

$$1 = \mathbb{E}^{\mu}[e^{\lambda b - (\lambda + \frac{\lambda^2}{2})T_b}].$$

Solve the equation $\alpha = \lambda + \frac{\lambda^2}{2} > 0$ to obtain (the ONLY positive) solution $\lambda = -\mu + \sqrt{\mu^2 + 2\alpha}$.

We, therefore, obtain, for $b > 0$ that (for any $\mu$) we have

$$\mathbb{E}^{\mu}[e^{-\alpha T_b}1_{\{T_b < \infty\}}] = e^{\mu b - b \sqrt{\mu^2 + 2\alpha}}, \alpha > 0.$$  

When $b < 0$, we have to do the same computations for $-W$ to obtain the result

$$\mathbb{E}^{\mu}[e^{-\alpha T_b}1_{\{T_b < \infty\}}] = e^{\mu b + b \sqrt{\mu^2 + 2\alpha}}, \alpha > 0.$$  

Problem 6.4. Using again the notations in Exercise 2, if $\mu > 0$:

$$\mathbb{P}[\sup_{0 \leq t < \infty} (-W_t) \in db] = 2\mu e^{-2\mu b} db, b > 0.$$  

Solution: Using the solution of the problem above, we let $\alpha \downarrow 0$ to obtain

$$\mathbb{P}^{\mu}(T_b < \infty) = e^{\mu b - |\mu| |b|},$$

for any $\mu$ and $b$. This is the probability of a Brownian motion with drift $\mu$ to reach level $b$ in finite time (done in class, already). Now, let $\mu > 0$. Under $P^{\mu}$, the process $-W$ is a BM with drift $-\mu$, so

$$\mathbb{P}^{\mu}(\sup_{0 \leq t < \infty} (-W_t) > b) = \mathbb{P}^{\mu}(T_b^W < \infty) = e^{(-\mu)b - |\mu| b} = e^{-2\mu b}, \forall b > 0,$$

according to the (interpretation of the) computation above. This finishes the solution.