

Examples of Elemental Proofs of Set Inclusion in Set Theory

The solutions to exercises #13, #14, and #15 of Section 6.2 are provided to illustrate how to write proofs of set inclusion in their fullest detail. Even though a proof of set inclusion will not have all of the detailed steps that are presented here actually written within the body of the proof, all of the detailed steps presented here should be going through the mind of the person reading a proof of set inclusion, filling in the details presented here which the less detailed proof left out.

In all of these proofs, pay attention to the fact that, within a direct proof of the original statement to be proven, there are direct proofs of intermediate statements to be proven (as in statements of subset relationships), and that within these direct proofs of intermediate statements to be proven, there are direct proofs of other statements to be proven (as in statements of containment in a set.) It is common to find a proof within a proof within a proof, and it is important, when reading and writing such proofs, to keep track of which statement is being proven at any point in the argument and to be sure that the proof of *that* statement is *complete* before continuing the proofs of other intermediate statements.

Section 6.2, Problem #13 (In Full Detail)

To Prove: For all sets A , B , and C , if $A \subseteq B$, then $(A \cap C) \subseteq (B \cap C)$.

Before presenting the proof, we list the definitions (in *procedural form*) that we will need.

Def'n of "subset of": $A \subseteq B \Leftrightarrow$ For all $x \in U$, if $x \in A$, then $x \in B$.

Def'n of "set intersection": For all $x \in U$, $x \in (A \cap B) \Leftrightarrow x \in A$ and $x \in B$.

We will also be using the following valid argument forms:

Specialization, Conjunction, Universal Modus Ponens.

Note: We write "Let $x \in X$ be given" to mean

"Let x be any given element of U such that $x \in X$."

Now for the proof.

To Prove: For all sets $A, B,$ and $C,$ if $A \subseteq B,$ then $(A \cap C) \subseteq (B \cap C).$

Proof:

[The following is a direct proof of the universal statement above.]

Let $A, B,$ and C be any sets.

Suppose $A \subseteq B.$

[$\therefore \forall x \in U,$ if $x \in A,$ then $x \in B,$ by definition of “subset of”.]

[We need to show that $(A \cap C) \subseteq (B \cap C).$

Recall that $(A \cap C) \subseteq (B \cap C) \Leftrightarrow \forall x \in U,$ if $x \in (A \cap C),$ then $x \in (B \cap C).]$

[The following is a direct proof proving, $\forall x \in U,$ if $x \in (A \cap C),$ then $x \in (B \cap C).]$

Let $x \in (A \cap C)$ be given. [We N.T.S. that $x \in (B \cap C).]$

$\therefore x \in A$ and $x \in C,$ by definition of “intersection (\cap)”. (***)

$\therefore x \in A,$ by specialization.

[“ $\forall y \in U, (y \in A) \rightarrow (y \in B)$ ” and “ $x \in A$ ” / “ $\therefore x \in B$ ” by Universal M. P. ”]

Since $A \subseteq B,$ $x \in B$ by Universal Modus Ponens.

Also, $x \in C,$ by specialization applied to (***) above.

$\therefore x \in B$ and $x \in C,$ by conjunction.

$\therefore x \in (B \cap C),$ by definition of “intersection (\cap)”.

[$\therefore \forall x \in U,$ if $x \in (A \cap C),$ then $x \in (B \cap C)]$

Therefore, $(A \cap C) \subseteq (B \cap C),$ by Direct Proof

Therefore, for all sets $A, B,$ and $C,$ if $A \subseteq B,$ then $(A \cap C) \subseteq (B \cap C)$ by Direct Proof.

Q E D

Section 6.2, Problem #14 (In Full Detail)

To Prove: For all sets A , B , and C , if $A \subseteq B$, then $(A \cup C) \subseteq (B \cup C)$.

Before presenting the proof, we list the definitions (in *procedural form*) that we will need.

Def'n of "subset of": $A \subseteq B \Leftrightarrow$ For all $x \in U$, if $x \in A$, then $x \in B$.

Def'n of "set union": For all $x \in U$, $x \in (A \cup B) \Leftrightarrow x \in A$ or $x \in B$.

We will also be using the following valid argument forms:

Generalization and Universal Modus Ponens.

Note: We write "Let $x \in X$ be given" to mean

"Let x be any given element of U such that $x \in X$."

Now for the proof.

To Prove: For all sets $A, B,$ and $C,$ if $A \subseteq B,$ then $(A \cup C) \subseteq (B \cup C).$

Proof:

[The following is a direct proof of the universal statement above.]

Let $A, B,$ and C be sets.

Suppose that $A \subseteq B.$

[$\therefore \forall x \in U,$ if $x \in A,$ then $x \in B,$ by definition of “subset of”.]

[We need to show that $(A \cup C) \subseteq (B \cup C).$

Recall that $(A \cup C) \subseteq (B \cup C) \Leftrightarrow \forall x \in U,$ if $x \in (A \cup C),$ then $x \in (B \cup C).]$

[The following is a direct proof proving, $\forall x \in U,$ if $x \in (A \cup C),$ then $x \in (B \cup C).]$

Let $x \in (A \cup C)$ be given. [We need to show that $x \in (B \cup C).]$

[This next statement is a list of cases to consider in this proof using “Division into Cases”.]

$\therefore x \in A$ or $x \in C,$ by definition of “union (\cup)”.

Case 1: ($x \in A$)

Suppose that $x \in A.$ [We need to show that $x \in (B \cup C).]$

Since $A \subseteq B,$ $x \in B$ by Universal Modus Ponens.

[An explanation of this use of Universal Modus Ponens appears in the previous proof.]

$\therefore x \in B$ or $x \in C,$ by Generalization.

$\therefore x \in (B \cup C)$ by definition of “union (\cup)”,
in the case that $x \in A.$ [End of Case 1.]

Case 2: ($x \in C$)

Suppose that $x \in C.$ [We need to show that $x \in (B \cup C).]$

$\therefore x \in B$ or $x \in C,$ by Generalization.

$\therefore x \in (B \cup C)$ by definition of “union (\cup)”,
in the case that $x \in C.$ [End of Case 2.]

Therefore, $x \in (B \cup C)$ in general .

[$\therefore \forall x \in U,$ if $x \in (A \cup C),$ then $x \in (B \cup C)]$

$\therefore (A \cup C) \subseteq (B \cup C),$ by Direct Proof .

\therefore For all sets $A, B,$ and $C,$ if $A \subseteq B,$ then $(A \cup C) \subseteq (B \cup C)$ by Direct Proof .

QED

Section 6.2, Problem #15 (In Full Detail)

To Prove: For all sets A and B , if $A \subseteq B$, then $B^C \subseteq A^C$.

Before presenting the proof, we list the definitions (in *procedural form*) that we will need.

Def'n of "subset of": $A \subseteq B \Leftrightarrow$ For all $x \in U$, if $x \in A$, then $x \in B$.

Def'n of "complement": For all $x \in U$, $x \in A^C \Leftrightarrow x \notin A$.

We will also be using the valid argument form: Universal Modus Tollens.

Note: We write "Let $x \in X$ be given" to mean

"Let x be any given element of U such that $x \in X$."

Now for the proof.

To Prove: For all sets A and B , if $A \subseteq B$, then $B^C \subseteq A^C$.

Proof:

[The following is a direct proof of the universal statement above.]

Let A and B be sets.

Suppose that $A \subseteq B$.

[$\therefore \forall x \in U$, if $x \in A$, then $x \in B$, by definition of "subset of".]

[We need to show that $B^C \subseteq A^C$.

Recall that $B^C \subseteq A^C \Leftrightarrow \forall x \in U$, if $x \in B^C$, then $x \in A^C$.]

[The following is a direct proof proving, $\forall x \in U$, if $x \in B^C$, then $x \in A^C$.]

Let $x \in B^C$ be given. [We need to show that $x \in A^C$.]

$\therefore x \notin B$, by definition of "complement".

[" $\forall y \in U$, $(y \in A) \rightarrow (y \in B)$ " and " $x \notin B$ " / " $\therefore x \notin A$ " by Universal M. T. "]

Since $A \subseteq B$, $x \notin A$ by Universal Modus Tollens.

$\therefore x \in A^C$, by definition of "complement".

[$\therefore \forall x \in U$, if $x \in B^C$, then $x \in A^C$.]

$\therefore B^C \subseteq A^C$, by Direct Proof.

\therefore For all sets A and B , if $A \subseteq B$, then $B^C \subseteq A^C$ by Direct Proof.

Q E D

To see how these proofs could be written without the full detail presented in the proofs above, read the proofs that follow. They are presented to show you what statements might be left out and what statements must remain. Please observe how sections of the proof are indented to indicate that a proof of an intermediate statement is being given within the proof of the original statement to be proven.

Note: We write “Let $x \in X$ be given” to mean
 “Let x be any given element of U such that $x \in X$.”

Solution to Problem #13 of Section 6.2. (Without the Full Detail)

To Prove: For all sets A, B , and C , if $A \subseteq B$, then $(A \cap C) \subseteq (B \cap C)$.

Proof:

Let A, B , and C be any sets.

Suppose $A \subseteq B$. [We N.T.S. that, $(A \cap C) \subseteq (B \cap C)$.]

Let $x \in (A \cap C)$ be given. [We N.T.S. that $x \in (B \cap C)$.]

$\therefore x \in A$ and $x \in C$, by definition of “intersection (\cap)”]

$\therefore x \in B$, since $A \subseteq B$ and $x \in A$.

$\therefore x \in (B \cap C)$, by definition of “intersection (\cap)”.

$\therefore (A \cap C) \subseteq (B \cap C)$ by Direct Proof.

Therefore, for all sets A, B , and C ,

if $A \subseteq B$, then $(A \cap C) \subseteq (B \cap C)$, by Direct Proof. **Q E D**

In the next proof, we use the fact that, for all sets X and Y ,

$X \subseteq (X \cup Y)$ and $Y \subseteq (X \cup Y)$, which is proved as follows:

To Prove: For all sets X and Y , $X \subseteq (X \cup Y)$ and $Y \subseteq (X \cup Y)$.

Proof: Let X and Y be any sets. [NTS: $X \subseteq (X \cup Y)$ and $Y \subseteq (X \cup Y)$]

Let $x \in X$ be given. Then, $x \in X$ or $x \in Y$, by generalization.

$\therefore x \in (X \cup Y)$ by definition of “union”.

$X \subseteq (X \cup Y)$, by Direct Proof.

Let $x \in Y$ be given. Then, $x \in X$ or $x \in Y$, by generalization.

$\therefore x \in (X \cup Y)$ by definition of “union”.

$Y \subseteq (X \cup Y)$, by Direct Proof.

$\therefore X \subseteq (X \cup Y)$ and $Y \subseteq (X \cup Y)$ by conjunction. **Q E D**

Solution to Problem #14 of Section 6.2. (Without the Full Detail)

To Prove: For all sets $A, B,$ and $C,$ if $A \subseteq B,$ then $(A \cup C) \subseteq (B \cup C).$

Proof: Let $A, B,$ and C be any sets.

Suppose that $A \subseteq B.$ [We N.T.S. that, $(A \cup C) \subseteq (B \cup C).]$

Let $x \in (A \cup C)$ be given. [We N.T.S. that $x \in (B \cup C).]$

$\therefore x \in A$ or $x \in C,$ by definition of “union (\cup)”.

Case 1: ($x \in A$) [We N.T.S. that $x \in (B \cup C).]$

Suppose $x \in A.$

Since $A \subseteq B,$ $x \in B,$ by Universal Modus Ponens.

$\therefore x \in (B \cup C)$ in Case 1, since $B \subseteq (B \cup C).$ [End of Case 1.]

Case 2: ($x \in C$) [We N.T.S. that $x \in (B \cup C).]$

Suppose $x \in C.$

$\therefore x \in (B \cup C)$ in Case 2, since $C \subseteq (B \cup C).$ [End of Case 2.]

Therefore, $x \in (B \cup C)$ in general.

$\therefore (A \cup C) \subseteq (B \cup C),$ by Direct Proof.

Therefore, for all sets $A, B,$ and $C,$ if $A \subseteq B,$ then $(A \cup C) \subseteq (B \cup C)$ by Direct Proof.

Q E D

Solution to Problem #15 of Section 6.2. (Without the Full Detail)

To Prove: For all sets A and $B,$ if $A \subseteq B,$ then $B^C \subseteq A^C.$

Proof:

Let A and B be sets such that $A \subseteq B.$ [We N.T.S. that $B^C \subseteq A^C.]$

Let $x \in B^C$ be given. [We N.T.S. that $x \in A^C.]$

$\therefore x \notin B,$ by definition of “complement”

Since $A \subseteq B$ and $x \notin B,$ $x \notin A,$ by Universal Modus Tollens.

$\therefore x \in A^C,$ by definition of “complement”.

$\therefore B^C \subseteq A^C,$ by Direct Proof.

\therefore For all sets A and $B,$ if $A \subseteq B,$ then $B^C \subseteq A^C,$ by Direct Proof.

Q E D