To Prove: For all integers \( n \), if \( n \) is odd, then \( n^2 \) is odd.

Proof: Let \( n \) be any integer.

Suppose \( n \) is odd.

\[ n = 2k + 1 \]

\[ n^2 = (2k + 1)^2 \]
\[ = 4k^2 + 4k + 1 \]
\[ = 2(2k^2 + 2k) + 1 \]

Let \( t = 2k^2 + 2k \), which is an integer since sums and products of integers are integers.

\[ n^2 = 2t + 1 \]

\[ n^2 \] is odd, by definition of "odd."

\[ n^2 \] is odd, by direct proof.

Q.E.D.
In applying the definition of "even" to \( m \), it was an error to use the variable "k" to represent the integer such that \( m \) equals 2 times it, because "k" had already been defined as the integer \( k \) such that \( n = 2k + 1 \).

A different variable, such as \( l \), should have been used, saying, for instance, "By definition of "even", \( m = 2l \) for some integer \( l \)."

42. In applying the definition of "even" to \( N \), it was an error to use the variable "k" to represent the integer such that \( N \) equals 2 times it, because "k" had already been defined as the integer \( k \) such that \( m = 2k \).

A different variable, such as \( l \), should have been used, saying, for instance, "By definition of "even", \( m = 2l \) for some integer \( l \)."
To Prove: The product of any even integer and any integer is even.

[Formal restatement: \( \forall m, n \in \mathbb{Z} \), if \( m \) is even, then \( mn \) is even.]

**Proof** Let \( m \) and \( n \) be any integers.

Suppose \( m \) is even.  \([\text{NTS: } \text{ } mn \text{ is even}]\)

\( m = 2k \) for some integer \( k \).

\[ mn = (2k)n \text{, by substitution,} \]
\[ = 2(kn) \text{, by rules of algebra.} \]

Let \( t = kn \), and \( t \) is an integer since a product of integers is an integer.

\[ mn = 2t \text{, by substitution.} \]

By definition of "even", \( mn \) is even.

The product of any even integer and any integer is even. \( Q.E.D. \)
#491 To Prove: The difference of any two odd integers is even.

[Formal restatement: \( \forall m, n \in \mathbb{Z}^{\text{odd}}, m - n \text{ is even.} \)]

**Proof:** Let \( m \) and \( n \) be any two odd integers.

[One could also say: "Let \( m \) and \( n \) be any integers.

Suppose \( m \) and \( n \) are both odd."]

By definition of "odd," there exist integers \( k \) and \( l \) such that \( m = 2k + 1 \) and \( n = 2l + 1 \).

\[
\begin{align*}
m - n &= (2k + 1) - (2l + 1) \\
&= 2k + 1 - 2l - 1 \\
&= 2k - 2l \\
&= 2(k - l) \\
&= 2t \quad \text{(all by rules of algebra, where } t = k - l, \text{ which is an integer since the difference of integers is an integer.)}
\end{align*}
\]

\[
\begin{align*}
m - n &= 2t \quad \text{by substitution,} \\
\text{m - n is even, by definition of "even."}
\end{align*}
\]

The difference of any two odd integers is even, by **DIRECT PROOF. Q.E.D.**