HW #4, SECTION 4.3 SOLUTIONS

Sec. 4.3:

3.) Yes. 5|0 because

\[ 0 = 5 \times 0 \]

5.) Yes. 4 \| 6m(2m+10)

\[
6m(2m+10) = 12m^2 + 60m
\]

\[ = 4(3m^2 + 15) \]

Let \( t = 3m^2 + 15 \), which is an integer.

\[ \therefore 6m(2m+10) = 4t \quad \text{so} \]

\[ 4 \mid 6m(2m+10) \]

\[ \text{The answer to the extra question is: } k = 3m^2 + 15 \]

12.) Yes. Let \( k \) be an integer and

let \( n = 4k+1 \). Then,

\[ n^2 - 1 = (4k+1)^2 - 1 = (16k^2 + 8k + 1) - 1 \]

\[ = 16k^2 + 8k \]

\[ = 8(2k^2 + k) \]

Let \( t = 2k^2 + k \). Then \( n^2 - 1 = 8t \), so \( 8 \mid (n^2 - 1) \)

\[ \text{The answer to the extra question is: } t = 2k^2 + k \]
Sec 4.3, #16.

To prove: For all integers \( a, b \) and \( c \), if \( a | b \) and \( a | c \), then \( a | (b - c) \).

Proof: Let \( a, b \) and \( c \) be any integers. Suppose \( a | b \) and \( a | c \).

By defn of "divides," there exist integers \( k \) and \( l \) such that \( b = ak \) and \( c = al \).

Then \( b - c = ak - al \) by substitution.

\[ = a(k - l) \text{ by rules of algebra.} \]

Let \( t = k - l \) which is an integer.

\[ \therefore b - c = at \text{ by substitution} \]

\[ \therefore a | (b - c) \text{ by definition of "divides."} \]

\[ \therefore \text{For all integers } a, b \text{, and } c, \text{ if } a | b \text{ and } a | c \text{, then } a | (b - c), \text{ by Direct Proof.} \]

#27) This is false. As a counterexample, let \( a = 5 \), \( b = 6 \) and \( c = 4 \);
then \( b + c = 10 \) and \( 5 | 10 \), so \( a | (b + c) \),
but \( 5 | 6 \) so \( a | b \) and \( 5 | 4 \), so \( a | c \).
SEC. 4,8, #28:

To Prove: The statement "For all integers \( a, b, \) and \( c \), 

\[ \text{if } a \mid b \text{, then } a \mid b \cdot a \cdot c \] 

is false.

Proof: [We will exhibit a counterexample.] 

Let \( a = 10 \), \( b = 2 \) and \( c = 5 \).

Then, \( bc = 10 \) and \( 10 = 10 \times 1 \).

\[ \therefore 10 \mid 10 \text{ and so } a \mid bc \text{, by substitution}. \]

Now, \( 10 > 2 \) and \( 10 > 5 \).

\[ \therefore \text{By Theorem 4.3.1 (and by Modus Tollens), } 10 \not\mid 2 \text{ at } 10 \not\mid 5. \]

\[ \therefore a \nmid b \text{ and } a \nmid c \text{ by substitution}. \]

\[ \therefore \text{With } a = 10, b = 2 \text{ and } c = 5, a \nmid bc \text{ and } a \nmid b \text{ and } a \nmid c. \]

\[ \therefore \text{The statement "For all integers } a, b, \text{ and } c, \]

\[ \text{if } a \mid bc \text{, then } a \mid b \cdot a \cdot c \] 

is false by proof - by - counter-example.

QED
Sec 4.3, #29, (NOT Assigned)

To Prove: For all integers $a$ and $b$, if $a | b$, then $a^2 \mid b^2$.

Proof: Let $a$ and $b$ be any integers such that $a \mid b$.

By definition of "divides," $b = a \cdot k$ for some integer $k$.

Let $l = k^2$, which is an integer because the product of integers is an integer.

Now, $b^2 = (a \cdot k)^2 = a^2 \cdot k^2$, by the rules of algebra.

Therefore, $b^2 = a^2 \cdot l$, by substitution.

$\therefore b^2 \mid a^2$, by definition of "divides."

For all integers $a$ and $b,$ if $a \mid b$, then $a^2 \mid b^2$, by direct proof.

$Q.E.D.$