TWO MORE EXAMPLES OF USING THE CHANGE OF VARIABLES METHOD WITH DOUBLE INTEGRALS

EXAMPLE 1: Let R be the region bounded by the trapezoid with vertices

$(2,0)$, $(4,0)$, $(0,8)$, $(0,4)$.

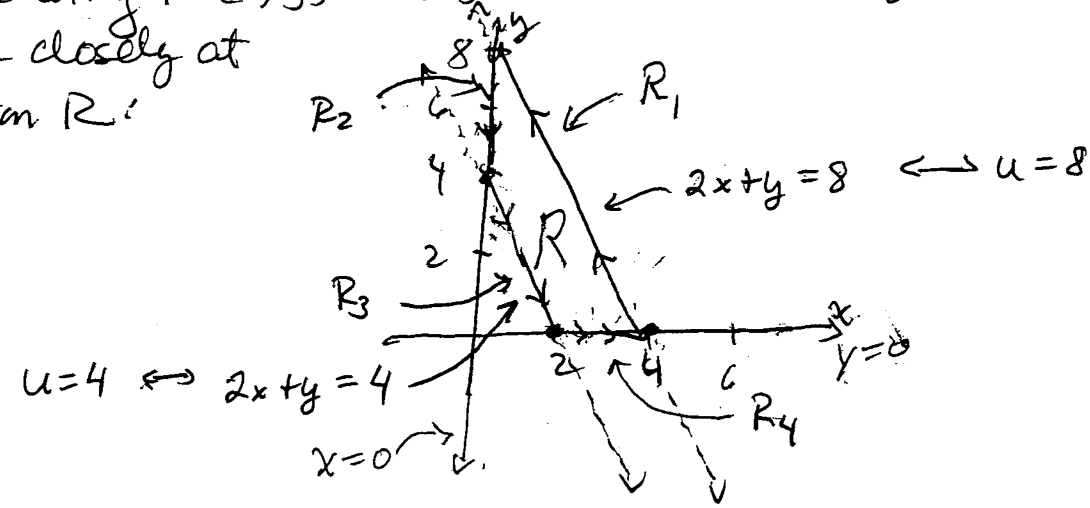
Show that $\iint_R \frac{2x-y}{2x+y}\, dA = 0$.

SOLUTION: We seek a region S in the u,v plane and a transformation T from S onto R.

We first define T^{-1} from the x,y plane to the u,v plane. Looking ahead to the possibility that the process will convert $\iint_R \frac{2x-y}{2x+y}\, dA$ to $\iint_S \frac{v}{u} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{(u,v)}$,

we consider letting $T^{-1}(x,y) = (u,v)$ where $u = 2x+y$ and $v = 2x-y$.

We look closely at Region R:



Considering that two sides of the trapezoid are $2x+y=4$ and $2x+y=8$, we are more confident in the choice to set $\underline{u=2x+y}$ and $\underline{v=2x-y}$.

We next determine formulas defining the transformation T. From the formulas for u and v above,

$$u+v=(2x+y)+(2x-y)=4x$$

$\underline{x=\frac{1}{4}(u+v)}$, and from $v=2x-y$, we have

$$y=2\left(\tfrac{1}{4}(u+v)\right)-v=\tfrac{1}{2}u+\tfrac{1}{2}v-v=\tfrac{1}{2}u-\tfrac{1}{2}v$$

$$\underline{y=\tfrac{1}{2}(u-v)}$$

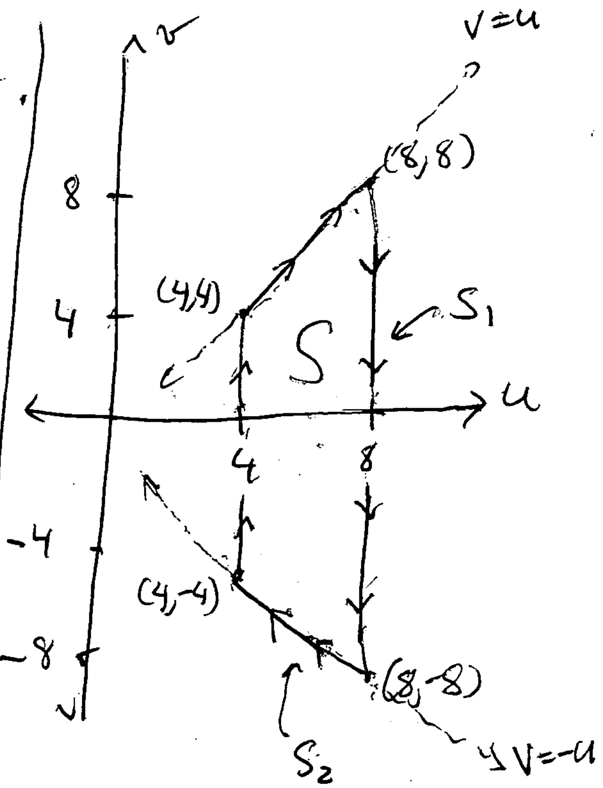
Let $T(u,v)=(x,y)=\left(\frac{1}{4}(u+v),\ \frac{1}{2}(u-v)\right)$.

Recall $T^{-1}(x,y)=(u,v)=(2x+y,\ 2x-y)$.

On R, R_1: $(4,0)\to(0,8)$ along $2x+y=8$.
$T^{-1}(4,0)=(8,8)$ and $T^{-1}(0,8)=(8,-8)$.
On S, S_1: $(8,8)\to(8,-8)$ along $u=8$

On R, R_2: $(0,8)\to(0,4)$ along $x=0$ and $4\le y\le 8$ and $T^{-1}(0,y)=(y,-y)$.
So on S, S_2: $(8,-8)\Rightarrow(4,-4)$ along $v=-u$.
You can also say, since $x=0$,
$\frac{1}{4}(u+v)=0\Rightarrow u+v=0\Rightarrow v=-u$.

Similarly,
S_3: $(4,-4)\to(4,4)$ along $u=4$
S_4: $(4,4)\to(8,8)$ along $v=u$.



Let S be the region in the u,v plane bounded by S_1, S_2, S_3 and S_4.

It can be shown that T maps S onto R.

Recall that $T(u,v) = (x,y)$
where $x = \frac{1}{4}(u+v)$
and $y = \frac{1}{2}(u-v)$.

The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{8} - \frac{1}{8} = -\frac{2}{8} = -\frac{1}{4}.$$

Its Absolute value is $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|-\frac{1}{4}\right| = \frac{1}{4}$.

Integrating, we get:

$$\iint_R \frac{2x-y}{2x+y}\,dA = \iint_S \frac{v}{u}\left(\frac{1}{4}\right)dA_{(u,v)} = \frac{1}{4}\int_4^8 \left(\int_{-u}^{u} \frac{v}{u}\,dv\right)du$$

$$= \frac{1}{4}\int_4^8 \left(\frac{1}{u}\left(\frac{1}{2}v^2\right)\Big]_{-u}^{u}\right)du$$

$$= \frac{1}{4}\int_4^8 \left(\frac{1}{2u}\cdot u^2 - \frac{1}{2u}(-u)^2\right)du$$

$$= \frac{1}{4}\int_4^8 \left(\frac{1}{2}u - \frac{1}{2}u\right)du = \frac{1}{4}\int_4^8 0\,du = 0,$$

Thus, we have shown that

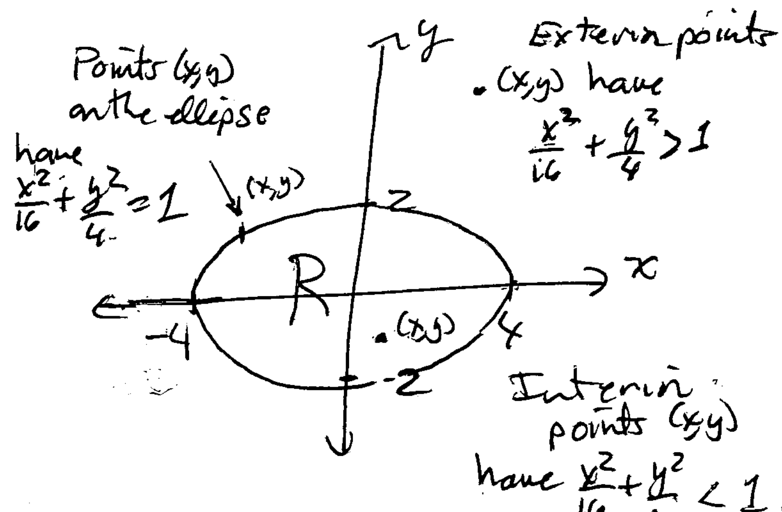
$$\iint_R \frac{x-y}{x+y}\,dA = 0,$$

EXAMPLE 2: FIND $\iint_R y^2 \, dA$ where

R is the region bounded by the ellipse
$4x^2 + 16y^2 = 64$.

Solution: In standard form, the ellipse equation is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$



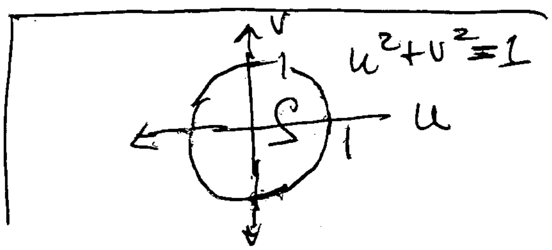
Another form is

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

Let $u = \frac{x}{4}$ and $v = \frac{y}{2}$

Then $x = 4u$ and $y = 2v$

Let $T(u,v) = (x,y)$ where $x = 4u$ and $y = 2v$.



Let S be the region in the u,v plane that is bounded by the circle $u^2+v^2=1$.

Suppose (u,v) is a point in the interior of the circle bounding S. Then $u^2+v^2<1$. Let $(x,y) = T(u,v)$. Then $x = 4u$ and $y = 2v$.

$\therefore \frac{x}{4} = u$ and $\frac{y}{2} = v$, so $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = u^2+v^2 < 1$.

$\therefore T(u,v) = (x,y)$ is a point in the interior of the ellipse

Similarly, [Recalling that $T(u,v) = (x,y) = (4u, 2v)$]

if (u,v) is a point on the circle bounding S, then

$T(u,v) = (x,y)$ is a point on the ellipse bounding R

and if (u,v) is a point outside the circle bounding S,

then $T(u,v) = (x,y)$ is a point exterior to the ellipse.

Thus T maps S onto R.

The Jacobian of T is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$

$$\therefore \iint_R y^2\,dA = \iint_S (2v)^2 (8)\,dA_{(u,v)} = 32 \iint_S v^2\,dA$$

[Setting $u = r\cos\theta$ and $v = r\sin\theta$]

$$= 32\int_0^{2\pi}\int_0^1 (r^2\sin^2\theta)(r)\,dr\,d\theta = 32\int_0^{2\pi}\int_0^1 r^3\sin^2\theta\,dr\,d\theta$$

$$= 32\int_0^{2\pi}\left(\frac{1}{4}r^4\sin^2\theta\right)\Bigg]_0^1 d\theta = 32\int_0^{2\pi}\frac{1}{4}\sin^2\theta\,d\theta$$

$$= 8\int_0^{2\pi}\sin^2 d\theta = 8\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = 8\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Bigg]_0^{2\pi}$$

$$= 8\left(\left(\frac{1}{2}2\pi - 0\right) - (0-0)\right) = 8\pi$$

So, $\iint_R y^2\,dA = 8\pi$