An Example of Using the
Change of Variables Method
in Double Integrals

Problem: Use a change of variables to find the volume of the solid region lying below the surface $z = f(x,y)$ and above the plane region R in the x,y plane, where

$$f(x,y) = (x+y)e^{(x-y)} \quad \text{and}$$

region R is the region bounded by the square with vertices $(4,0)$, $(6,2)$, $(4,4)$, $(2,2)$.

SOLUTION: The STEPS to accomplish are:

STEP ①: FIND A CONVENIENT REGION S in the u,v plane and an appropriate transformation $(x,y) = T(u,v)$ which maps S onto R.

Here, $T(u,v) = (x,y)$ with $x = g(u,v)$ and $y = h(u,v)$.

Note: In performing this step, we first construct the Inverse of T, $(u,v) = T^{-1}(x,y)$ with $u = G(x,y)$ and $v = H(x,y)$. Then, we derive the formulas defining $(u,v) = T(x,y)$ by solving for x and y in terms of u and v from the formulas defining $T^{-1}(x,y)$.

STEP (2) : Compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$ for the

transformation T and compute its absolute value, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

STEP (3) : Apply the CHANGE OF VARIABLES FORMULA:

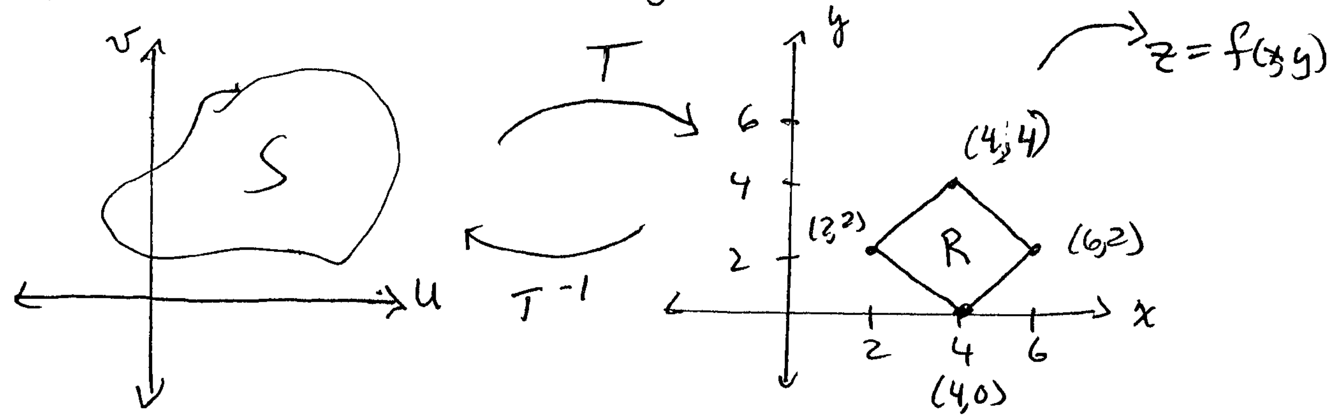
$$\iint_R f(x,y)\,dA_{(x,y)} = \iint_S f\big(g(u,v), h(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{(u,v)}$$

Note: When evaluating the integral on the right, it may be necessary to express the integral using polar coordinates

$u = r\cos\theta$, $v = r\sin\theta$, $u^2+v^2=r^2$.

PERFORMING THE STEPS:

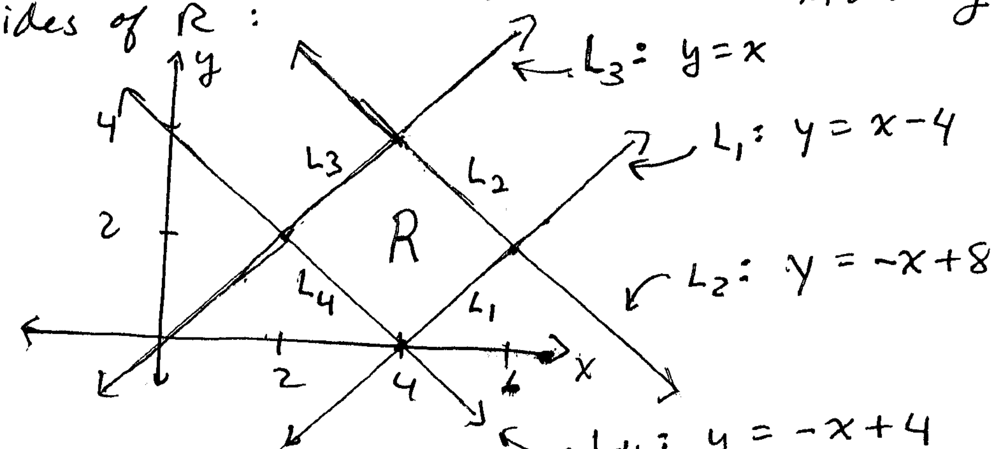
STEP ①: First plot Region R:



Also, look at the Integrand

$$f(x,y) = (x+y)\,e^{(x-y)}.$$

Next, look at the equations of the lines extending the sides of R:



$$\left.\begin{aligned} L_1:\ & y = x - 4 \Rightarrow y - x = -4 \Rightarrow x - y = 4 \\ L_3:\ & y = x \Rightarrow y - x = 0 \Rightarrow x - y = 0 \end{aligned}\right\}\ \text{Consider setting } v = x - y$$

$$\left.\begin{aligned} L_2:\ & y = -x + 8 \Rightarrow x + y = 8 \\ L_4:\ & y = -x + 4 \Rightarrow x + y = 4 \end{aligned}\right\}\ \text{Consider setting } u = x + y$$

Continuing Step ①: Make it official. Define $T^{-1}(x,y) = (u,v)$.

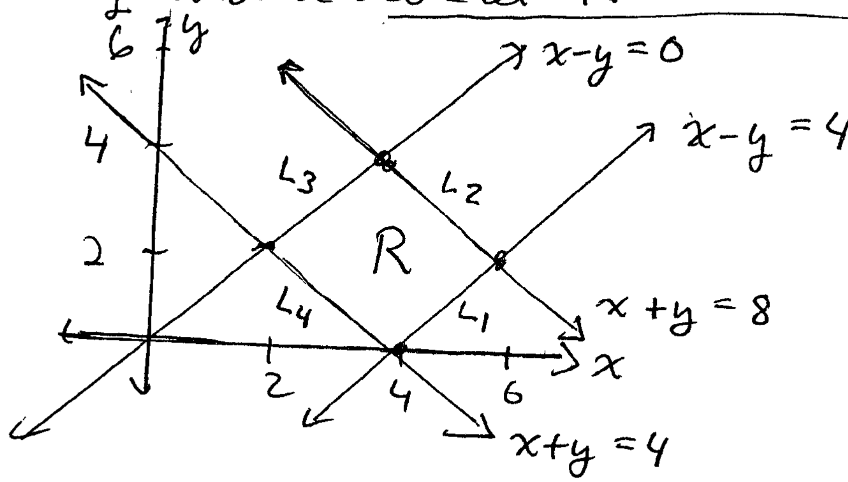
Let $T^{-1}(x,y) = (u,v)$ where

$u = x+y = G(x,y)$ and

$v = x-y = H(x,y)$.

The Next two tasks: (A) Discover Region S in the u,v plane and (B) Derive the formulas for $T(u,v) = (x,y)$.

(A) Discover region Region S in the u,v plane by taking another look at R:



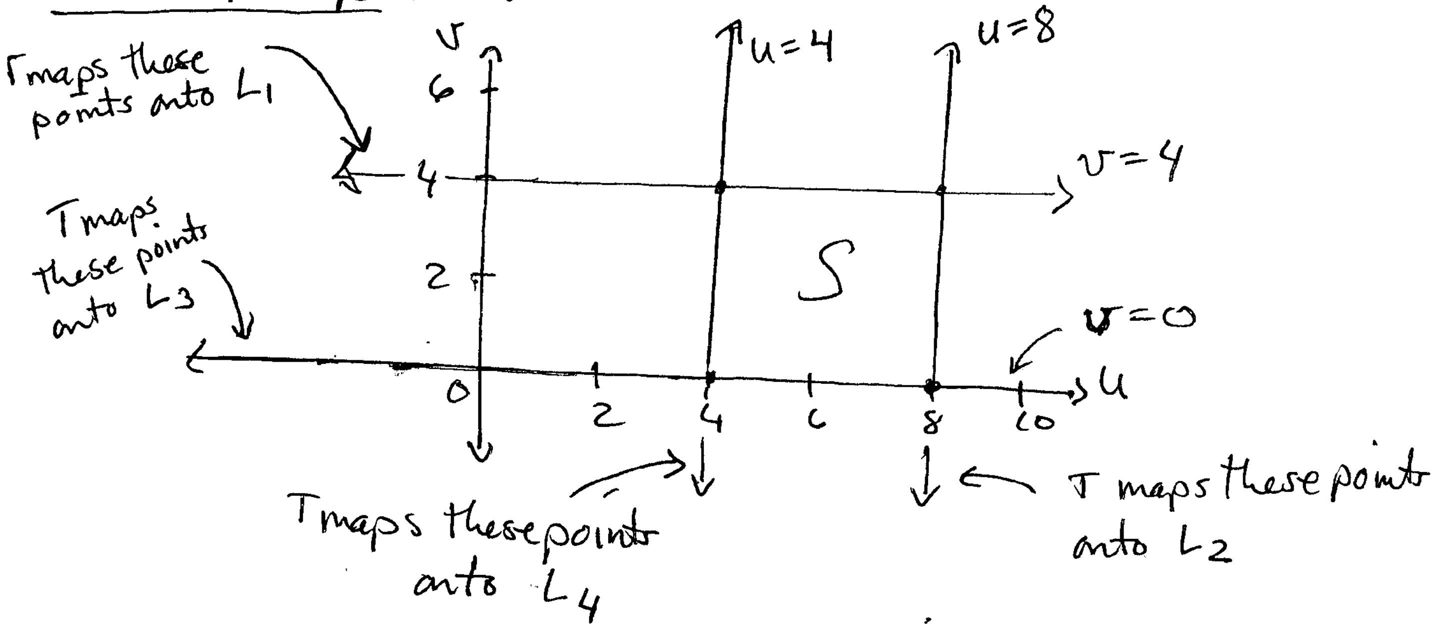
For points (x,y) on L_1, $x-y=4$,
so, $T^{-1}(x,y) = (u,v)$, where $v=4$ and u varies, so for these points (u,v) on the horizontal line $v=4$, $T(u,v)$ is on L_1.

For points (x,y) on L_2, $x+y=8$,
so, $T^{-1}(x,y) = (u,v)$, where $u=8$ and v varies, so for these points (u,v) on the vertical line $u=8$, $T(u,v)$ is on L_2.

For points (x,y) on L_3, $x-y=0$,
so, $T^{-1}(x,y)=(u,v)$, where $v=0$ and u varies,
so, for these points (u,v) on the horizonal line $v=0$,
$T(u,v)$ is on L_3.

For points (x,y) on L_4, $x+y=4$,
so, $T^{-1}(x,y)=(u,v)$, where $u=4$ and v varies, so
for these points (u,v) on the vertical line $u=4$,
$T(u,v)$ is on L_4.

So, T maps the horizonal line $v=4$ onto the line L_1 in the xy plane,
and T maps the vertical line $u=8$ onto the line L_2 in the xy plane,
and T maps the horizontal line $v=0$ onto the line L_3 in the xy plane,
and T maps the vertical line $u=4$ onto the line L_4 in the xy plane.



Let region S be the bounded region demarcated by
the lines $v=4$, $u=8$, $v=0$ and $u=4$; That is,
S is the square region bounded by these lines in the u,v plane.

These lines in the u,v plane demarcate several regions of the u,v plane, but only one region, region S, is bounded. The other regions demarcated by these lines are all unbounded.

The linearity of transformation T assures us that T maps the bounded region S onto the bounded region of the x,y plane, demarcated by those lines in the xy plane onto which T maps the lines in the u,v plane that bound S;

That is, Transformation T maps Region S of the u,v plane onto region R of the x,y plane; since T maps the unbounded regions in the u,v plane onto unbounded regions in the x,y plane.

Since T maps S onto R and we can use transformation T in the Change of Variables Formula.

(B) Deriving the formulas for $(x,y) = T(u,v)$.

We seek $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$.

Recall, $T^{-1}(x,y) = (u,v)$ where $\underline{u = x+y}$ and $\underline{v = x-y}$.

We solve for x and y in terms of u and v:

(Isolating x:) $u+v = (x+y)+(x-y) = 2x$

so, $\boxed{x = \frac{1}{2}(u+v) = g(u,v)}$

(Solving for y:) From $u = x+y$ and $x = \frac{1}{2}(u+v)$,

$u = \frac{1}{2}(u+v) + y \Rightarrow y = u - \frac{1}{2}u - \frac{1}{2}v = \frac{1}{2}u - \frac{1}{2}v$.

So, $\boxed{y = \frac{1}{2}(u-v) = h(u,v)}$

So, we define

$$T(u,v) = (x,y) \text{ where } x = \tfrac{1}{2}(u+v)$$
$$\text{and } y = \tfrac{1}{2}(u-v).$$

STEP ① IS FINISHED.

STEP ② Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ for Transformation T and also compute $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial y} = \frac{1}{2}, \quad \frac{\partial y}{\partial u} = \frac{1}{2}, \quad \frac{\partial y}{\partial v} = -\frac{1}{2}.$$

So,
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

and so, $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$ and its absolute value is

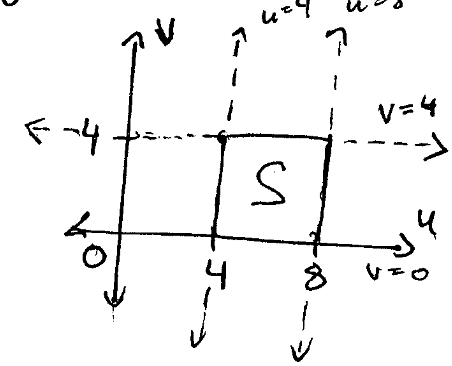
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}.$$

Note: It is typical that the Jacobian is not a constant as it is here, but rather it is a function of the two variables u and v.

STEP ③ Apply the CHANGE OF VARIABLES FORMULA:

$$\iint_R f(x,y)\,dA_{(x,y)} = \iint_S f(\underbrace{x(u,v)}_{x=g(u,v)}, \underbrace{y(u,v)}_{y=h(u,v)}) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{(u,v)}.$$

Here: $\iint_R (x+y)e^{(x-y)}\,dA_{(x,y)}$



(by (*) below) $\overset{\searrow}{=} \iint_S (ue^v)(\tfrac{1}{2})\,dA_{(u,v)}$

$$= \int_4^8 \int_0^4 \tfrac{1}{2}ue^v\,dv\,du \leftarrow \text{Fix } u$$

$$= \int_4^8 \left[\tfrac{1}{2}ue^v\right]_{0=v}^{4=v} du = \int_4^8 \left(\tfrac{1}{2}ue^4 - \tfrac{1}{2}ue^0\right) du$$

$$= \int_4^8 \left(\frac{e^4}{2}u - \tfrac{1}{2}u\right) du = \int_4^8 \left(\frac{e^4-1}{2}\right) u\,du$$

$$= \left(\frac{e^4-1}{2}\right) \int_4^8 u\,du = \left(\frac{e^4-1}{2}\right)\left[\tfrac{1}{2}u^2\right]_4^8$$

$$= \left(\frac{e^4-1}{2}\right)(32-8) = \left(\frac{e^4-1}{2}\right)(24) = \underline{12(e^4-1)}$$

So, $\boxed{\iint_R (x+y)e^{(x-y)}\,dA = 12(e^4-1)}$

DONE!

FOOTNOTE (*) Appears on the next page.

(*) By definition,

$T(u,v) = (x,y)$, where

$$x = x(u,v) = \tfrac{1}{2}(u+v)$$

and

$$y = y(u,v) = \tfrac{1}{2}(u-v).$$

(Note: The fact that these functions were derived from the formulas $u = x+y$ and $v = x-y$ suggests that we can replace $x+y$ by u and we can replace $x-y$ by v, when we replace $f(x,y)$ by $f(x(u,v), y(u,v))$;

But, we will prove these replacements rigorously.)

By definition of T,

$$x+y = \tfrac{1}{2}(u+v) + \tfrac{1}{2}(u-v) = \tfrac{1}{2}u + \tfrac{1}{2}v + \tfrac{1}{2}u - \tfrac{1}{2}v$$

So, $x+y = u$.

Also, $x-y = \tfrac{1}{2}(u+v) - \tfrac{1}{2}(u-v) = \tfrac{1}{2}u + \tfrac{1}{2}v - \tfrac{1}{2}u + \tfrac{1}{2}v$

so, $x-y = v$.