MORE NOTES ON DOUBLE INTEGRALS OVER MORE GENERAL REGIONS

Let $z = f(x, y)$ be given. Suppose D is a *closed and bounded* region in the xy plane that is contained in the domain of f. Let R be a rectangle $[a, b] \times [c, d]$ that contains the region D.

Dr. Shirley's Definition of the Double Integral over region D

$$\iint_D f(x,y)\, dA := \lim_{m,n\to\infty} \left(\sum_i \sum_j f\left(x_{ij}^*, y_{ij}^*\right) \Delta A_{ij} \right)$$

where we only sum over those selected points $\left(x_{ij}^*, y_{ij}^*\right)$ such that $\left(x_{ij}^*, y_{ij}^*\right)$ is in region D.

How do we determine what number this is?

Double Riemann Sums. Here, the Double Riemann Sum $R_{m,n}$ is

$$R_{m,n} = \sum_i \sum_j f\left(x_{ij}^*, y_{ij}^*\right) \Delta A_{ij}$$

but only for those selected points $\left(x_{ij}^*, y_{ij}^*\right)$ such that $\left(x_{ij}^*, y_{ij}^*\right)$ is in region D.

Imagine a general region D inside the rectangle R. R is subdivided into $m \times n$ smaller rectangular regions, and there is a randomly placed point $\left(x_{ij}^*, y_{ij}^*\right)$ inside each subrectangle. There are four cases:

(1) If the subrectangle does not intersect D at all, we do not include that $\left(x_{ij}^*, y_{ij}^*\right)$.
(2) If the subrectangle is a subset of D, we do include that $\left(x_{ij}^*, y_{ij}^*\right)$.
(3) If the subrectangle partially intersects the region D, and the point $\left(x_{ij}^*, y_{ij}^*\right) \in D$, then we do include $\left(x_{ij}^*, y_{ij}^*\right)$.
(4) If the subrectangle partially intersects the region D, and the point $\left(x_{ij}^*, y_{ij}^*\right) \notin D$, then we do not include $\left(x_{ij}^*, y_{ij}^*\right)$.

We can use iterated integrals when the region D is a a Type I Region or Type II Region, as defined below.

Type I Region. Generally speaking, a Type I Region has an upper bound of a curve $y = g_2(x)$ and a lower bound of a curve $y = g_1(x)$, and between those two curves, all points where the x coordinate is in a particular interval $[a, b]$ are included. In other words:

A *Type I Region* has the form

$$D = \{(x, y) \text{ such that } a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

In this case, when D is a Type I Region,

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Type II Region. Generally speaking, a Type II Region has a left bound of a curve $x = h_1(y)$ and a right bound of a curve $x = h_2(y)$, and between those two curves, all points where the y coordinate is in a particular interval $[c, d]$ are included. In other words:

A *Type II Region* has the form

$$D = \{(x, y) \text{ such that } c \leq y \leq c, h_1(y) \leq x \leq h_2(y)\}.$$

In this case, when D is a Type II Region,

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Examples

Type I Example. Determine $\iint_D (x + 2y)\, dA$ where D is the region bounded by $y = 2x^2$ and $y = 1 + x^2$.

Solution. The two curves intersect at the points $(-1, 2)$ and $(1, 2)$, and between those two points, the parabolas enclose a Type I region in which $y = 1 + x^2$ is the upper bound and $y = 2x^2$ is the lower bound. In other words:

$$D = \left\{(x, y) | -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\right\}$$

$$\begin{aligned}
\iint_D (x+2y)\,dA &= \int_{-1}^{1}\int_{2x^2}^{1+x^2} \,dy\,dx \\
&= \int_{-1}^{1} \left((xy+y^2)\Big|_{y=2x^2}^{y=1+x^2} \right)\,dx \\
&= \int_{-1}^{1} \left(\left(x(1+x^2)+(1+x^2)^2\right) - \left(x(2x^2)+(2x^2)^2\right)\right)\,dx \\
&= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx \\
&= \left(-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \right)\Big|_{-1}^{1} \\
&= \frac{32}{15}
\end{aligned}$$

Therefore:

$$\iint_D (x+2y)\,dA = \frac{32}{15} = 2\frac{2}{15}$$

Type II Example. Determine $\iint_D (xy)\,dA$ where D is the region bounded by $x = \frac{y^2}{2} - 3$ and $x = y+1$.

Solution. The two curves intersect at the points $(-1,-2)$ and $(5,4)$. The enclosed region R has the parabola as it's left bound and the line as its right bound. Therefore:

$$D = \left\{ (x,y) | -2 \le y \le 4, \left(\frac{y^2}{2} - 3\right) \le x \le (y+1) \right\}$$

We can set up the integral as follows:

$$\iint_D (xy)\,dA = \int_{-2}^{4} \left(\int_{x=\frac{y^2}{2}-3}^{y+1} xy\,dx \right) dy$$

The computations are left to the reader.

A Property of the Double Integral

In closing, note that if the region D has an unusual shape, we can sometimes break it into pieces, each of which we can use iterated integration on. If $D = D_1 \cup D_2$, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$