

How the Multi-Variable Chain Rule (Case 1 Version) follows from the Differential dz

Given a function $z = f(x, y)$ of two variables where $x = x(t)$ and $y = y(t)$ are also functions of t , a change in t values, say by Δt , produces a change Δx in x -value and a change Δy in y -value, which ultimately causes a change Δz in z -value.

The Total Differential Principle says that

$$\Delta z \approx dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} dy.$$

In any application, $dx = \Delta x$ and $dy = \Delta y$.

$$\text{So, } \Delta z \approx dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y.$$

$$\text{Now, } \frac{dz}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left(\frac{1}{\Delta t} \right) \cdot \Delta z.$$

Using the differential approximation of Δz by dz ,
These limits then are equal:

$$\frac{dz}{dt} = \lim_{\Delta t \rightarrow 0} \left(\frac{1}{\Delta t} \right) \Delta z = \lim_{\Delta t \rightarrow 0} \left(\frac{1}{\Delta t} \right) \left(\frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \Delta y \right)$$

$$\text{So, } \frac{dz}{dt} = \lim_{\Delta t \rightarrow 0} \left(\frac{\partial z}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \cdot \frac{\Delta y}{\Delta t} \right) \quad (2)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \left(\lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \right) + \frac{\partial z}{\partial y} \left(\lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} \right)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$
