The Two Problems, presented here with worked-out Solutions, were in my notes for the lecture for Monday, Nov 11, 2013.

The First PROBLEM WAS PRESENTED AT THE END OF CLASS, BUT TIME DID NOT ALLOW ME TO PRESENT THE COMPLICATED ALGEBRA REQUIRED TO FIND The Solution. THAT COMPLICATED ALGEBRA IS PRESENTED HERE.

THE SECOND PROBLEM IS AN EXAMPLE OF A PROBLEM TO FIND The Point on a graph which is closest to a given fixed point.

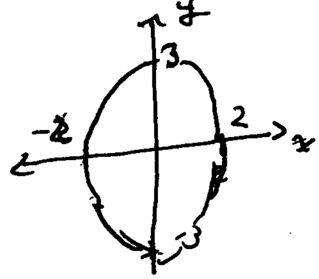
Dro F. Shirly.

Problem: Let $z = f(x,y) = xy^2$.

Determine the max. value of f and the min. value of f subject to the constraint $9x^2 + 4y^2 = 36$.

Sol'n: Maximize / minimize

$f(x,y) = xy^2$ subject to $g(x,y) = 9x^2 + 4y^2 - 36 = 0$



$\nabla f = \langle y^2, 2xy \rangle$; $\nabla g = \langle 18x, 8y \rangle$

Find all points (x,y) so that a number λ exist such that $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 0$.

Evaluate $f(x,y)$ at all of these points and the largest is the max and the smallest is the min.

$\nabla f = \lambda \nabla g \Rightarrow \langle y^2, 2xy \rangle = \lambda \langle 18x, 8y \rangle$

We must solve three equations simultaneously:

① $y^2 = \lambda(18x) = (18x)\lambda$

② $2xy = \lambda(8y)$

③ $9x^2 + 4y^2 = 36$ [Do not forget to include the constraint equation!]

We would like to be able to divide by y in ②, so we must use cases!

Case 1

Assume $y = 0$

By ③, $9x^2 = 36$

$x^2 = 4$

$x = \pm 2$, $y = 0$

Points: $(2,0)$, $(-2,0)$

$f(2,0) = f(-2,0) = 0$

Case 2

Assume $y \neq 0$. [Divide ② by y]

By ②, $2x = 8\lambda \Rightarrow x = 4\lambda \Rightarrow x^2 = 16\lambda^2$

Also, with $x = 4\lambda$, by ①, $y^2 = (18\lambda)(4\lambda) = 72\lambda^2$

By ③, $9(16\lambda^2) + 4(72\lambda^2) = 36$ [$\div 36$]

$\Rightarrow 4\lambda^2 + 8\lambda^2 = 1 \Rightarrow 12\lambda^2 = 1$

$\Rightarrow \lambda^2 = \frac{1}{12} = \frac{3}{36} \Rightarrow \boxed{\lambda = \pm \frac{\sqrt{3}}{6}}$

Case 2 : $y \neq 0$ (cont.)

Recall $y = \pm \frac{\sqrt{3}}{6}$ and $x = 4\lambda$ and $y^2 = 72\lambda^2$

So, $x = \pm \frac{4\sqrt{3}}{6} \Rightarrow x = \pm \frac{2\sqrt{3}}{3}$

and $y^2 = 72\left(\frac{3}{36}\right) = 6 \Rightarrow y = \pm\sqrt{6}$

$\Rightarrow (x,y) = \left(\pm\frac{2\sqrt{3}}{3}, \pm\sqrt{6}\right)$ 4 points!

With $f(x,y) = xy^2$,

$f\left(\frac{2\sqrt{3}}{3}, \sqrt{6}\right) = \frac{2\sqrt{3}}{3} \times 6 = 4\sqrt{3} \approx 6.93$

$f\left(\frac{2\sqrt{3}}{3}, -\sqrt{6}\right) = \frac{2\sqrt{3}}{3} \times 6 = 4\sqrt{3}$

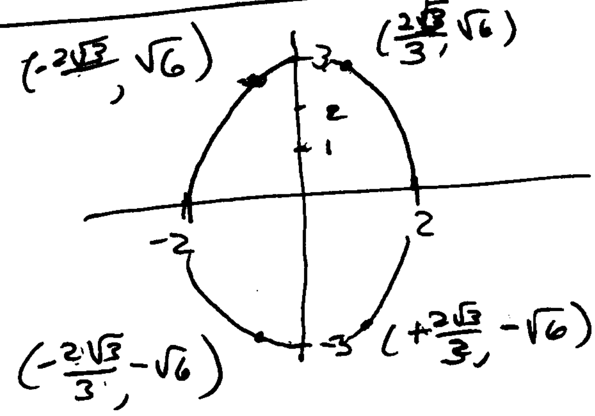
$f\left(-\frac{2\sqrt{3}}{3}, \sqrt{6}\right) = \frac{-2\sqrt{3}}{3} \cdot 6 = -4\sqrt{3} \approx -6.93$

$f\left(-\frac{2\sqrt{3}}{3}, -\sqrt{6}\right) = \frac{-2\sqrt{3}}{3} \times 6 = -4\sqrt{3}$

also $\left.\begin{array}{l} f(-2,0) = 0 \\ f(2,0) = 0 \end{array}\right\}$ from Case 1.

The maximum value of f subject to the constraint $g(x,y) = 0$ is $4\sqrt{3} = f\left(\frac{2\sqrt{3}}{3}, \sqrt{6}\right)$.

The minimum value of f subject to the constraint $g(x,y) = 0$ is $-4\sqrt{3} = f\left(-\frac{2\sqrt{3}}{3}, \sqrt{6}\right)$



$\frac{2\sqrt{3}}{3} \approx 0.577$

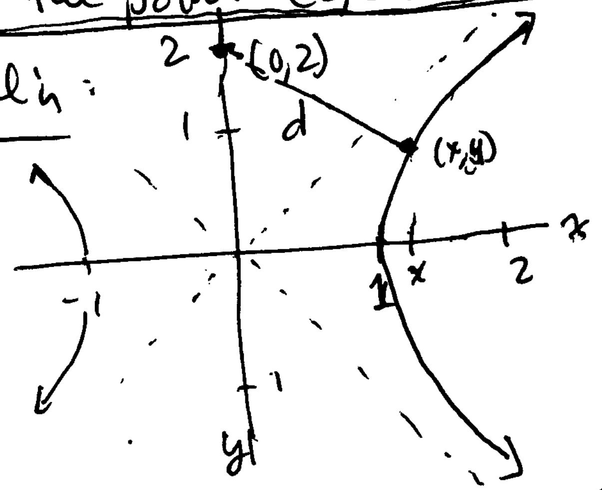
$\sqrt{6} \approx 2.449$

$f\left(\frac{2\sqrt{3}}{3}, \sqrt{6}\right) = 4\sqrt{3}$

$f\left(-\frac{2\sqrt{3}}{3}, \sqrt{6}\right) = -4\sqrt{3}$

<u>Problem</u>: FIND the point on the hyperbola branch $x^2 - y^2 = 1$, $x \geq 0$, which is closest to the point $(0,2)$.

<u>Sol'n</u>:



Minimize $d = \sqrt{x^2 + (y-2)^2}$

Minimize
$$d^2 = f(x,y) = x^2 + (y-2)^2$$
on $g(x,y) = x^2 - y^2 - 1 = 0$
$$x \geq 0$$

Since there is no MAX value of $f(x,y) = d^2$ and only one point (x_0, y_0) with minimum d, we will probably find only one point with a number λ so that $\nabla f = \lambda \nabla g$.

$$f(x,y) = x^2 + (y-2)^2 \qquad g(x,y) = x^2 - y^2 - 1$$

$$\nabla f = \langle 2x, 2(y-2) \rangle \; ; \quad \nabla g = \langle 2x, -2y \rangle$$

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2(y-2) \rangle = \lambda \langle 2x, -2y \rangle$$

Solve

① $2x = \lambda 2x$

② $2(y-2) = \lambda(-2y)$

③ $x^2 - y^2 = 1$

[Don't forget to include The Constraint Equation ③]

By ③, $x \neq 0$;
So, by dividing by $2x$ in ①, $\lambda = 1$
By ②, with $\lambda = 1$, $2y - 4 = -2y \Rightarrow 4y = 4$,

So, $y = 1$. $\Rightarrow$ By ③, $x^2 - 1 = 1 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Since $x \geq 0$, $x = \sqrt{2}$. The only point with minimum d^2 is $(x_0, y_0) = (\sqrt{2}, 1)$

The point on the branch of the hyperbola $x^2 - y^2 = 1$ which is closest to the point $(0,2)$ is $(x_0, y_0) = (\sqrt{2}, 1)$.