

PROBLEMS FOR "LAST DAY REVIEW"

Problem #1: Approximate the following definite integral correct to three decimal places:

$$\int_0^1 x \cos(x^3) dx$$

Sol'n: Since $f(x) = x \cos(x^3)$ does not have a formula for its antiderivative, we must represent $f(x) = x \cos(x^3)$ with a power series.

THREE STEPS

- ① Get a PSR for $x \cos(x^3)$
- ② Represent $s = \int_0^1 x \cos(x^3) dx$ with a power series.
- ③ Approximate s with a partial sum s_n .

$$(1) \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad R = \infty$$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} \quad (x^3)^{2n} = x^{6n}$$

$$f(x) = x \cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!} \quad x \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} C_n x^{n+1}$$

$$\begin{aligned} (2) \quad s &= \int_0^1 x \cos(x^3) dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!} \right) dx = \sum_{n=0}^{\infty} \left(\int_0^1 \frac{(-1)^n x^{6n+1}}{(2n)!} dx \right) \\ &= \sum_{n=0}^{\infty} \left((-1)^n \cdot \frac{1}{(2n)!} \cdot \frac{x^{6n+2}}{(6n+2)} \right) \Big|_0^1 \\ &= \sum_{n=0}^{\infty} \left((-1)^n \cdot \frac{1}{(2n)!} \cdot \frac{1}{(6n+2)} \right) \end{aligned}$$

Problem
#1 (Cont.)

$$S = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \frac{1}{(6n+2)}$$

$$1^{6n+2} = 1 \quad (2)$$

(3)

$$S = b_0 - b_1 + b_2 - b_3 + \dots \quad b_k = \frac{1}{(2k)!(6k+2)}$$

$$\text{For } k=0, \quad b_0 = \frac{1}{2}$$

$$\text{For } k=1, \quad b_1 = \frac{1}{2 \times 8} = \frac{1}{16} = 0.0625$$

$$\text{For } k=2, \quad b_2 = \frac{1}{(4!)(14)} = \frac{1}{336} = 0.002976$$

$$\text{For } k=3,$$

$$b_3 = \frac{1}{(6!)(20)} = \frac{1}{14,400} = 0.000069 \leq \underline{\underline{0.0005}}$$

$$|\text{Error in } S_n \text{ vs } S| \leq b_{n+1} = b_3 \quad \begin{matrix} n+1=3 \\ n=2 \end{matrix}$$

$$S_2 = b_0 - b_1 + b_2 = \frac{1}{2} - \frac{1}{16} + \frac{1}{336}$$

$$S_2 \approx 0.44047619$$

$$S = \int_0^1 x \cos(x^3) dx \approx \underline{\underline{0.440}}$$

Problem #2: Given the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1} = \sum a_n,$$

Using the RATIO Test, Determine the Radius of Convergence R and determine the Interval of Convergence.

Solution:

Absolute value! $\rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-3)^{n+1}}{2(n+1)+1} \times \frac{2n+1}{(x-3)^n} \right|$

$$= \frac{|x-3|^{n+1}}{|x-3|^n} \times \frac{2n+1}{2n+3}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = |x-3| \times \frac{2n+1}{2n+3}$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} |x-3| \cdot \frac{2n+1}{2n+3} = |x-3| \cdot \frac{2}{2} = |x-3|$$

$$L = |x-3| < 1$$

So, Radius $R = 1$

Since $|x-3| < 1$,

$$-1 < x-3 < 1$$

+3 +3 +3

Adding 3:

$$2 < x < 4 = \text{The Interior of the I.O.C.}$$

Note: If L had been
 $L = 4|x-3| < 1$,
Then we need to divide
by 4 first to get
 $|x-3| < \frac{1}{4}$ and
we would have $R = \frac{1}{4}$

We next need to check for convergence at the endpoints $x=2$ and $x=4$.

We must set $x=2$ and $x=4$ in the original power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$$

Problem #2 (continued)

(4)

$$\text{At } x=2: \sum a_n = \sum_{n=0}^{\infty} (-1)^n \frac{(2-3)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{2n+1}$$

(Series Diverges!)

$$= \sum_{n=0}^{\infty} \frac{((-1)(-1))^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1^n}{2n+1}$$

So, $\sum a_n = \sum_{n=0}^{\infty} \frac{1}{2n+1}$, which can be shown to diverge.

$\sum_{n=0}^{\infty} \frac{1}{2n+1}$ Diverges by the Limit Comparison Test applied to the Harmonic Series $\sum_{n=0}^{\infty} \frac{1}{n}$ which diverges.

Also $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ Diverges by the direct comparison test with $\sum_{n=0}^{\infty} \frac{1}{2n} = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{1}{n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n}$ which also diverges.

At $x=4$: $\sum a_n = \sum_{n=0}^{\infty} (-1)^n \frac{(4-3)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$

(Series Converges!)

which is an alternating series such that

①: $\frac{1}{2(n+1)+1} = \frac{1}{2n+3} < \frac{1}{2n+1}$ for all $n \geq 1$

and

②: $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{2n+1} = 0$,

So the alternating series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ Converges by the Alternating Series Test.

$\therefore$ The Radius of Convergence is $R=1$

The Interval of Convergence is $(2, 4] = \{x \in \mathbb{R} \mid 2 < x \leq 4\}$

Bonus Problem 3

(5)

For the function $f(x) = \frac{1}{x^3}$,

- (1) Find the "n=2" Taylor Polynomial $T_2(x)$ centered at $a=1$,
and

- (2) Estimate the error in the approximation

$$T_2(1.1) \approx \frac{1}{(1.1)^3}, \text{ that is, at } x=1.1.$$

Solution: $a=1$

①	$\frac{n}{n=0}$	$\frac{f^{(n)}(x)}{\frac{1}{x^3} = x^{-3}}$	$\frac{f^{(n)}(a) = f^{(n)}(1)}{\frac{1}{1^3} = 1}$	$\frac{f^{(n)}(1)}{n!}$
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$n=1$ $-3x^{-4} = \frac{-3}{x^4}$ $\frac{-3}{1^4} = -3$

$n=2$ $12x^{-5} = \frac{12}{x^5}$ $\frac{12}{1^5} = 12$

$n=3$ $-60x^{-6} = \frac{-60}{x^6} = f^{(3)}(x)$

$n=4$ $360/x^7 = f^{(4)}(x)$

$T_2(x) = \underline{\hspace{2cm}}?$

$$T_2(x) = \frac{1}{0!} + \frac{(-3)}{1!}(x-1) + \frac{12}{2!}(x-1)^2$$

$$T_2(x) = 1 - 3(x-1) + 6(x-1)^2$$

- (2) We use Taylor's Inequality (which will be provided if needed)

Find "d": $\begin{array}{c} 1-d \quad x \quad 1+d \\ \leftarrow \quad \quad \rightarrow \\ 0.9 \quad 1 \quad 1.1 \end{array}$

$$0.9 \leq x \leq 1.1 \Rightarrow (0.9-1) \leq x-1 \leq (1.1-1)$$

$$\Rightarrow -0.1 \leq x-1 \leq 0.1 \Rightarrow |x-1| \leq 0.1 = d$$

Taylor's Inequality:

$$\left| \text{ERROR in } T_2(1.1) \approx f(1.1) \right| = |R_2(1.1)| \leq \frac{M}{3!} |1.1-1.0|^3 = \frac{M}{3!} (0.1)^3$$

$$\text{Here } M \geq \max |f^{(3)}(x)| \text{ for } 0.9 \leq x \leq 1.1 \quad (4)$$

$$M \geq \max \left| \frac{-60}{x^6} \right| \text{ for } 0.9 \leq x \leq 1.1$$

For $n=4$, $f^{(4)}(x) = \frac{360}{x^7} \neq 0$, so $f^{(3)}(x)$ has no critical points and no local max and no local min in the interval of $[0.9, 1.1]$.

So $\max \left| \frac{-60}{x^6} \right|$ occurs at $x=0.9$ or at $x=1.1$.

$$\left. \frac{60}{x^6} \right|_{x=0.9} = 112.9 \text{ and } \left. \frac{60}{x^6} \right|_{x=1.1} = 33.9$$

$$\text{Set } \underline{M = 112.9} \text{ and set } d = 0.1$$

$$|R_2(1.1)| \leq \frac{112.9}{3!} (1.1 - 1.0)^3 \approx (18.817)(0.1)^3$$

$$\boxed{\left| \text{ERROR in } T_2(1.1) \approx \frac{1}{(1.1)^3} \right| \leq 0.018817} \leftarrow \text{Solution!}$$

In fact,

$$T_2(1.1) = 1 - 3(0.1) + 6(0.1)^2 = 1 - 0.3 + 0.06 = \underline{\underline{0.7600}}$$

and

$$\frac{1}{(1.1)^3} \approx 0.7513, \text{ so Actual error is}$$

$$\left| \frac{1}{(1.1)^3} - T_2(1.1) \right| = \underline{\underline{0.0087}} < 0.018817$$