Problem #1: Approximate the following definite integral correct to three decimal places:

$$\int_0^1 x\cos(x^3)\,dx$$

Sol'n: Since $f(x) = x\cos(x^3)$ does not have a formula for its antiderivative, we must represent $f(x) = x\cos(x^3)$ with a power series.

THREE STEPS

① Get a PSR for $x\cos(x^3)$

② Represent $s = \int_0^1 x\cos(x^3)\,dx$ with a power series.

③ Approximate s with a partial sum s_n.

(1)

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad R = \infty$$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} \qquad (x^3)^{2n} = x^{6n}$$

$$f(x) = x\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(6n+1)}}{(2n)!} \qquad x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$$

(2)

$$s = \int_0^1 x\cos(x^3)\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}\right)dx = \sum_{n=0}^{\infty}\left(\int_0^1 \frac{(-1)^n x^{6n+1}}{(2n)!}\,dx\right)$$

$$= \sum_{n=0}^{\infty}\left((-1)^n \frac{1}{(2n)!}\cdot\frac{x^{6n+2}}{(6n+2)}\right)\Bigg|_0^1$$

$= 0$

Problem #1 (Cont.) $S = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \frac{1}{(6n+2)}$ $\quad 1^{6n+2} = 1$ (2

(3) $S = b_0 - b_1 + b_2 - b_3 + \cdots - \qquad b_k = \frac{1}{(2k)!(6k+2)}$

For $k=0$, $b_0 = \frac{1}{2}$

For $k=1$, $b_1 = \frac{1}{2\times 8} = \frac{1}{16} = 0.0625$

For $k=2$, $b_2 = \frac{1}{(4!)(14)} = \frac{1}{336} = 0.002976$

24×14

For $k=3$,

$b_3 = \frac{1}{14{,}400} = 0.000069$

$(6!)(20)$

$\underline{\underline{\leq 0.0005}}$

$|\text{Error in } s_n \approx s| \leq b_{n+1} = b_3 \qquad n+1=3$

$\underline{\underline{n=2}}$

$S_2 = b_0 - b_1 + b_2 = \frac{1}{2} - \frac{1}{16} + \frac{1}{336}$

$S_2 \approx 0.44047619$

$S = \int_0^1 x\cos(x^3)\,dx \approx \underline{\underline{0.440}}$

PROBLEM #2: Given the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1} = \sum a_n ,$$

Using the RATIO Test, Determine the RADIUS of Convergence R and determine the Interval of Convergence.

SOLUTION:

Absolute value!

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{(x-3)^n}\right|$$

$$= \frac{|x-3|^{n+1}}{|x-3|^n} \cdot \frac{2n+1}{2n+3}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = |x-3| \cdot \frac{2n+1}{2n+3}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x-3| \cdot \frac{\cancel{n}(2+\frac{1}{n})}{\cancel{n}(2+\frac{3}{n})} = |x-3| \cdot \frac{2}{2}$$

$$L = |x-3| < 1$$

So, Radius $R = 1$

Note: If L had been $L = 4|x-3| < 1$, Then we need to divide by 4 first to get $|x-3| < \frac{1}{4}$ and we would have $R = \frac{1}{4}$

Since $|x-3| < 1$,

$$\begin{array}{ccccc} -1 & < & x-3 & < & 1 \\ +3 & & +3 & & +3 \end{array}$$

Adding 3:

$2 < x < 4$ = The Interior of the I.O.C.

We next need to check for convergence at the endpoints $x = 2$ and $x = 4$.

We must set $x = 2$ and $x = 4$ in the original power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1} .$$

Problem #2 (continued)

At $x=2$: (Series Diverges!) $\sum a_n = \sum_{n=0}^{\infty} (-1)^n \frac{\overset{(2-3)^n}{(-1)^n}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{2n+1}$

$$= \sum_{n=0}^{\infty} \frac{((-1)(-1))^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1^n}{2n+1}$$

So, $\sum a_n = \sum_{n=0}^{\infty} \frac{1}{2n+1}$, which can be shown to diverge.

$\sum_{n=0}^{\infty} \frac{1}{2n+1}$ Diverges by the Limit Comparison Test applied to the Harmonic Series $\sum_{n=0}^{\infty} \frac{1}{n}$ which diverges.

Also $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ Diverges by the direct comparison test with $\sum_{n=0}^{\infty} \frac{1}{2n} = \sum_{n=0}^{\infty} \frac{1}{2}\cdot\frac{1}{n} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n}$ which also diverges.

At $x=4$: (Series Converges!) $\sum a_n = \sum_{n=0}^{\infty} (-1)^n \frac{\overset{(4-3)^n}{1^n}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$

which is an alternating series such that

① : $\frac{1}{2(n+1)+1} = \frac{1}{2n+3} < \frac{1}{2n+1}$ for all $n \geq 1$

and

② $\lim b_n = \lim_{n \to \infty} \frac{1}{2n+1} = 0$,

So the alternating series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ Converges by the Alternating Series Test.

$$\boxed{\begin{array}{l} \therefore \text{The Radius of Convergence is } R = 1 \\ \text{The Interval of Convergence is } (2,4] = \{x \in \mathbb{R} \mid 2 < x \leq 4\} \end{array}}$$

Bonus Problem #3

For the function $f(x) = \frac{1}{x^3}$,

① Find the "$n=2$" Taylor Polynomial $T_2(x)$ centered at $a=1$,

and

② Estimate the error in the approximation $T_2(1.1) \approx \frac{1}{(1.1)^3}$, that is, at $x = 1.1$.

Solution: $a = 1$

①

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(1)$
$n=0$	$\frac{1}{x^3} = x^{-3}$	$\frac{1}{1^3} = 1$
$n=1$	$-3x^{-4} = \frac{-3}{x^4}$	$\frac{-3}{1^4} = -3$
$n=2$	$12x^{-5} = \frac{12}{x^5}$	$\frac{12}{1^5} = 12$
$n=3$	$-60x^{-6} = \frac{-60}{x^6} = f^{(3)}(x)$	
$n=4$	$360/x^7 = f^{(4)}(x)$	

$T_2(x) = $ ____ ?

$$T_2(x) = \frac{1}{0!} + \frac{(-3)}{1!}(x-1) + \frac{12}{2!}(x-1)^2$$

$$\boxed{T_2(x) = 1 - 3(x-1) + 6(x-1)^2}$$

② We use Taylor's Inequality (which will be provided if needed)

FIND "d":

$1-d \quad x \quad 1+d$

$0.9 \quad 1 \quad 1.1$

$0.9 \le x \le 1.1 \Rightarrow (0.9-1) \le x-1 \le (1.1-1)$

$\Rightarrow -0.1 \le x-1 \le 0.1 \Rightarrow \underline{|x-1| \le 0.1 = d}$

Taylor's Inequality:

$$\left| \text{ERROR in } T_2(1.1) \approx f(1.1) \right| = |R_2(1.1)| \le \frac{M}{3!}|1.1 - 1.0|^3 = \frac{M}{3!}(0.1)^3$$

Here $M \geq \max |f^{(3)}(x)|$ for $0.9 \leq x \leq 1.1$

$M \geq \max \left| \frac{-60}{x^6} \right|$ for $0.9 \leq x \leq 1.1$

For $n=4$, $f^{(4)}(x) = \frac{360}{x^7} \neq 0$, so $f^{(3)}(x)$ has no critical points and no local max and no local min in the interior of $[0.9, 1.1]$.

So $\max \left| \frac{-60}{x^6} \right|$ occurs at $x = 0.9$ or at $x = 1.1$.

$\left. \frac{60}{x^6} \right|_{x=0.9} = 112.9$ and $\left. \frac{60}{x^6} \right|_{x=1.1} = 33.9$.

Set $\underline{M = 112.9}$ and set $d = 0.1$

$|R_2(1.1)| \leq \frac{112.9}{3!}(1.1-1.0)^3 \approx (18.817)(0.1)^3$

$\boxed{\left| \text{ERROR in } T_2(1.1) \approx \frac{1}{(1.1)^3} \right| \leq \underline{0.018817}}$ ← Solution!

In fact,

$T_2(1.1) = 1 - 3(0.1) + 6(0.1)^2 = 1 - 0.3 + 0.06$
$= \underline{0.7600}$

and

$\frac{1}{(1.1)^3} \approx 0.7513$, So Actual ERROR is

$\left| \frac{1}{(1.1)^3} - T_2(1.1) \right| = \underline{0.0087 < 0.018817}$