Integrating Powers and Products of Trig functions **Review of Techniques**

In all of the formulas below, m, n, k, and t are all positive integers .

I. Strategy for Integrating $\int (\sin^m x)(\cos^n x)\,dx$

A. If n is odd,
use u = sin x , du = cos x dx and write all the $\cos^2 x$ **factors as** $\left(1-\sin^2 x\right)$**.**

B. If m is odd,
use u = cos x , du = -sin x dx and write all the $\sin^2 x$ **factors as** $\left(1-\cos^2 x\right)$**.**

C. If both m and n are even, then
1) if m = n , use sin 2x = 2 (sin x)(cos x) to simplify the integrand.

2) otherwise, write all the $\sin^2 x$ **factors as** $\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)$**,**

and then write all the $\cos^2 x$ **factors as** $\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)$**.**

Examples

Type I A:

$$\int (\sin^6 x)(\cos^3 x)\,dx = \int (\sin^6 x)(\cos^2 x)(\cos x)\,dx = \int (\sin^6 x)(1-\sin^2 x)(\cos x)\,dx = \int u^6 (1-u^2)\,du$$

Type I B:

$$\int (\sin^5 x)(\cos^2 x)\,dx = \int (\sin^4 x)(\cos^2 x)(\sin x)\,dx = \int (1-\cos^2 x)^2 (\cos^2 x)(\sin x)\,dx = -\int (1-u^2)^2 u^2\,du$$

Type I C:

$$\int (\sin^4 x)(\cos^4 x)\,dx = \int (\sin x \cos x)^4\,dx = \int \left(\frac{1}{2}\sin 2x\right)^4 dx = \frac{1}{16}\int (\sin^2 2x)^2\,dx = \frac{1}{16}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)^2 dx$$

II. Strategy for Integrating $\int (\tan^m x)(\sec^n x)\,dx$

A. If n is even,
use u = tan x , du = $\sec^2$ x dx and write all the other $\sec^2 x$ factors as $\left(1+\tan^2 x\right)$.

B. If m is odd,
use u = sec x , du = (sec x) (tan x) dx and write all the $\tan^2 x$ factors as $\left(\sec^2 x - 1\right)$.

C. Otherwise, try using trig identities, integration by parts, etc.
--- Nothing is definite here.

Examples

Type II A:

$$\int (\tan^8 x)(\sec^4 x)\,dx = \int (\sec^2 x)(\tan^8 x)(\sec^2 x)\,dx = \int (\tan^2 x + 1)(\tan^8 x)(\sec^2 x)dx = \int (u^2+1)\;u^8\,du$$

Type II B:

$$\int (\tan^3 x)(\sec^7 x)\,dx = \int (\sec^6 x)(\tan^2 x)\;(\sec x\;\tan x)\,dx = \int (\sec^6 x)(\sec^2 x - 1)(\sec x\;\tan x)\,dx = \int u^6\,(u^2-1)\;du$$

Type II C:

$$\int (\sec^7 x)(\tan^4 x)\,dx = \int (\sec^7 x)(\tan^2 x)^2\,dx = \int (\sec^7 x)(\sec^2 x - 1)^2\,dx =$$

$$\int (\sec^7 x)(1 - 2\sec^2 x + \sec^4 x)\,dx = \int \sec^7 x\,dx - 2\int \sec^9 x\,dx + \int \sec^{11} x\,dx$$

From this point, one can apply the techniques for integrating single powers of trig functions (Integrals of Type III).

III. Strategy for Integrating:

A. $\int \sin^m x\, dx$ **,** **B.** $\int \cos^n x\, dx$ **,** **C.** $\int \tan^k x\, dx$ **,** **D.** $\int \sec^t x\, dx$

A. If m is odd, use the methods for Type I B.

If m is even, write all the $\sin^2 x$ **factors as** $\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)$**.**

B. If n is odd, use the methods for Type I A.

If n is even, write all the $\cos^2 x$ **factors as** $\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)$**.**

C. If k is odd, then save one (tan x) factor and write all the $\tan^2 x$ **factors as** $\left(\sec^2 x - 1\right)$**, and then use the methods for Type II B.**

If k is even, then use the reduction formula:

$$\int \tan^k x\, dx = \frac{1}{k-1}\tan^{k-1} x - \int \tan^{k-2} x\, dx$$

D. If t is even, then use the methods for Type II A.

If t is odd, then use the reduction formula:

$$\int \sec^t x\, dx = \frac{1}{t-1}\tan x\, \sec^{t-2} x + \frac{t-2}{t-1}\int \sec^{t-2} x\, dx$$

The following two integrals are also used in the above methods:

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \textbf{and} \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

They are derived as follows:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C = \ln|\cos x|^{-1} + C = \ln|\sec x| + C$$

$$\int \sec x\, dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx = \int\left(\frac{\sec x\ \tan x + \sec^2 x}{\sec x + \tan x}\right) dx = \int \frac{1}{u}\, du = \ln|\sec x + \tan x| + C$$

Useful Trigonometric Identities:

The Pythagorean Identity:

$$\cos^2 x + \sin^2 x = 1 \;, \qquad \textit{which implies}: \; \cos^2 x = 1 - \sin^2 x \;, \; \sin^2 x = 1 - \cos^2 x$$

The Double Angle Formulas (derived from the sum-of-angle formulas):

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x \;, \qquad \textit{which implies}: \; \cos 2x = 1 - 2\sin^2 x \;, \; \textit{and} \;\; \cos 2x = 2\cos^2 x - 1$$

The sec ^{2} x and tan ^{2} x Formulas:

$$\sec^2 x = 1 + \tan^2 x \qquad \textit{and} \qquad \tan^2 x = \sec^2 x - 1$$

The very useful cos ^{2} x and sin ^{2} x Formulas:

$$\cos^2 x = \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) \qquad \textit{and} \qquad \sin^2 x = \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right)$$

Derivations:

Divide both sides of $\cos^2 x + \sin^2 x = 1$ **by** $\cos^2 x$ **;** **Result:** $1 + \tan^2 x = \sec^2 x$

This immediately gives $\sec^2 x = 1 + \tan^2 x$ *and* $\tan^2 x = \sec^2 x - 1$ **.**

To both sides of $\cos 2x = 2\cos^2 x - 1$ **, add 1 ;** **Result:** $2\cos^2 x = 1 + \cos 2x$

Now, multiply by ½ ; Result: $\cos^2 x = \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right)$

To both sides of $\cos 2x = 1 - 2\sin^2 x$ **,**

add $2\sin^2 x$ **and subtract** $\cos 2x$ **;** **Result:** $2\sin^2 x = 1 - \cos 2x$

Now, multiply by ½ ; Result: $\sin^2 x = \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right)$