

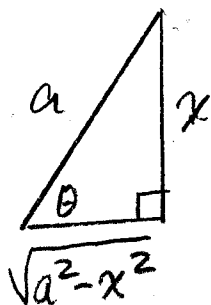
TRIG SUBSTITUTION Principles

Let $a > 0$ be a constant

Use this method when the Integrand involves one of the following expressions:

$$\sqrt{a^2 - x^2}$$

DRAW:



$$\frac{x}{a} = \sin \theta$$

Substitutions:

$$x = a \sin \theta$$

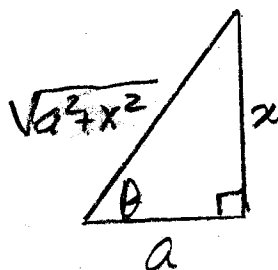
$$dx = a \cos \theta d\theta$$

$$\sqrt{a^2 - x^2} = a \cos \theta$$

$$\theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\sqrt{a^2 + x^2}$$

DRAW:



$$\frac{x}{a} = \tan \theta$$

Substitutions:

$$x = a \tan \theta$$

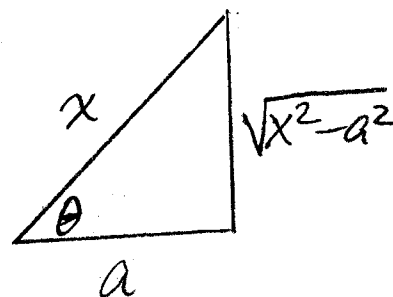
$$dx = a \sec^2 \theta d\theta$$

$$\sqrt{a^2 + x^2} = a \sec \theta$$

$$\theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\sqrt{x^2 - a^2}$$

DRAW:



$$\frac{x}{a} = \sec \theta$$

Substitutions:

$$x = a \sec \theta$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2 - a^2} = a \tan \theta$$

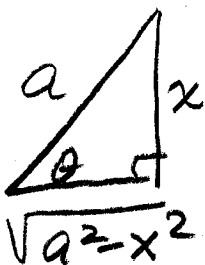
$$\theta = \sec^{-1}\left(\frac{x}{a}\right)$$

TRIG SUBSTITUTION EXAMPLES

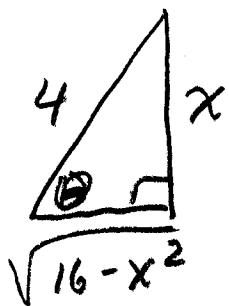
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EXAMPLE 1 : FIND $\int \frac{1}{(16-x^2)^{3/2}} dx$

$$\int \frac{1}{(16-x^2)^{3/2}} dx = \int \frac{1}{(\sqrt{16-x^2})^3} dx$$

$a = 4, \quad \sqrt{16-x^2} = \sqrt{a^2-x^2}$ draw 

Here:



$$\frac{x}{4} = \sin \theta \Rightarrow x = 4 \sin \theta$$

$$dx = 4 \cos \theta d\theta$$

$$\frac{\sqrt{16-x^2}}{4} = \frac{\text{Adj}}{\text{Hyp}} = \cos \theta \Rightarrow \sqrt{16-x^2} = 4 \cos \theta$$

$$\int \frac{1}{(\sqrt{16-x^2})^3} dx = \int \frac{1}{(4 \cos \theta)^3} (4 \cos \theta) d\theta$$

$$= \int \frac{4 \cos \theta}{64 \cos^3 \theta} d\theta = \frac{1}{16} \int \frac{1}{\cos^2 \theta} d\theta =$$

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Ex 1 (continued)

3

$$= \frac{1}{16} \int \sec^2 \theta d\theta = \frac{1}{16} \tan \theta + C$$

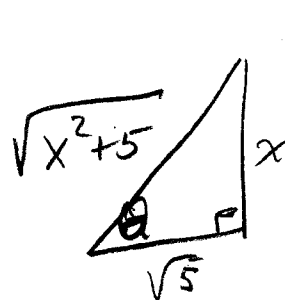
(Referring back to the triangle drawn earlier)

$$= \frac{1}{16} \frac{x}{\sqrt{16-x^2}} + C = \frac{x}{16\sqrt{16-x^2}} + C$$

EXAMPLE 2

$$\int \left(\frac{1}{\sqrt{x^2+5}} \right)^3 dx = \textcircled{D}$$

$a = \sqrt{5}$



$$\begin{aligned} \frac{x}{\sqrt{5}} &= \tan \theta \\ x &= \sqrt{5} \tan \theta \\ dx &= \sqrt{5} \sec^2 \theta d\theta \end{aligned}$$

$$\frac{\sqrt{x^2+5}}{\sqrt{5}} = \sec \theta \Rightarrow \sqrt{x^2+5} = \sqrt{5} \sec \theta$$

$$\textcircled{D} = \int \left(\frac{1}{\sqrt{5} \sec \theta} \right)^3 (\sqrt{5} \sec^2 \theta d\theta) = \int \frac{\sqrt{5} \sec^2 \theta}{5 \sqrt{5} \sec^3 \theta} d\theta$$

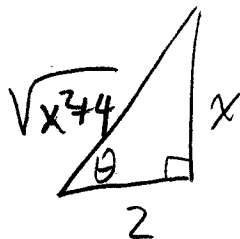
$$= \frac{1}{5} \int \frac{1}{\sec \theta} d\theta = \frac{1}{5} \int \cos \theta d\theta = \frac{1}{5} \sin \theta + C$$

$$= \boxed{\frac{1}{5} \cdot \frac{x}{\sqrt{x^2+5}} + C}$$

Example 3

4

$$\int \frac{x}{\sqrt{x^2+4}} dx$$



$$a=2$$

$$\frac{x}{2} = \tan \theta \Rightarrow x = 2 \tan \theta \Rightarrow dx = 2 \sec^2 \theta d\theta$$

$$\frac{\sqrt{x^2+4}}{2} = \sec \theta \Rightarrow \sqrt{x^2+4} = 2 \sec \theta$$

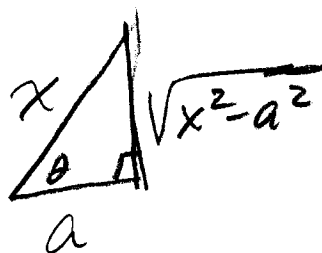
$$\int \frac{x}{\sqrt{x^2+4}} dx = \int \frac{2 \tan \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta = 2 \int \sec \theta \tan \theta d\theta$$

$$= 2 \sec \theta + C = 2 \cdot \frac{\sqrt{x^2+4}}{2} + C = \sqrt{x^2+4} + C$$

Note: EXAMPLE 3 could be solved more easily using straight u-substitution.

Example 4

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \textcircled{E}$$



$$\frac{x}{a} = \sec \theta$$

$$x = a \sec \theta$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$\frac{\sqrt{x^2-a^2}}{a} = \tan \theta \Rightarrow \sqrt{x^2-a^2} = a \tan \theta$$

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EXAMPLE 4 (continued)

$$\textcircled{E} = \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{a \tan \theta} a \sec \theta \tan \theta d\theta$$

$$= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$

$$= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C$$

$$= \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + C$$

$$= \ln |x + \sqrt{x^2 - a^2}| - \ln a + C$$

$$= \boxed{\ln |x + \sqrt{x^2 - a^2}| + C}$$
