Applying the Integral Test Correctly and

Suggested Wording of the Required Justifications for the Integral Test

The Integral Test applies to a series \( \sum_{n=1}^{\infty} a_n \) such that there exists a function \( y = f(x) \) which satisfies

the **FOUR SPECIAL CONDITIONS**:

1. Function \( f \) is continuous on \([1, \infty)\) (or on \([K, \infty)\) for some \(K \geq 1\)),
2. Function \( f \) is positive (that is, \( f(x) \geq 0 \)) on \([1, \infty)\) (or on \([K, \infty)\) for some \(K \geq 1\)),
3. \( f(n) = a_n \) for all \(n \geq 1\) (for all \(n \geq K\) for some \(K \geq 1\)),
4. Function \( f \) is decreasing on \([1, \infty)\) (or on \([K, \infty)\) for some \(K \geq 1\)).

The **FIRST TASK** for BOTH CASES, the Case of Convergence and the Case of Divergence:

The FIRST TASK in applying the integral test is to STATE that Special Conditions (1), (2), and (3) are true about Function \( f \), and also to STATE and to VERIFY that Special Condition (4) is true about Function \( f \).

For Special Conditions (1), (2), and (3), it is enough only to state "Function \( f \) is continuous and positive on \([1, \infty)\) and \( f(n) = a_n \) for all \(n \geq 1\)."

The verification that Special Condition (4) is true about Function \( f \), that is, that Function \( f \) is decreasing on \([1, \infty)\), can be accomplished algebraically by proving that, whenever \(1 \leq x_1 \leq x_2\), \( f(x_1) \geq f(x_2) \).

However, Special Condition (4) is frequently verified by showing that the derivative \( f' \) is negative (that is, \( f'(x) < 0 \)) on \([1, \infty)\) (or on \([K, \infty)\) for some \(K \geq 1\)).

**Sentences and Explanations concerning the Special Conditions (1), (2), (3), and (4) are REQUIRED.**

After these sentences and explanations have been provided, you must then state:

"The Integral Test Applies."

Then, after stating "The Integral Test Applies," you must evaluate the correct improper integral

\[
\int_{1}^{\infty} f(x) \, dx.
\]
The Final Task in Applying the Integral Test: Writing a Required Justification

CASE 1: The improper integral $\int_{1}^{\infty} f(x) \, dx$ is CONVERGENT:

In this case, you must write a justification as clear and complete as the following:

"Since the integral $\int_{1}^{\infty} f(x) \, dx$ is Convergent, 
the series $\sum_{n=1}^{\infty} a_n$ is Convergent by the Integral Test."

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CASE 2: The improper integral $\int_{1}^{\infty} f(x) \, dx$ is DIVERGENT:

In this case, you must write a justification as clear and complete as the following:

"Since the integral $\int_{1}^{\infty} f(x) \, dx$ is Divergent, 
the series $\sum_{n=1}^{\infty} a_n$ is Divergent by the Integral Test."

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On the next page is an example of a complete and correct presentation of an application of the Integral Test.
FOR EXAMPLE: Consider the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \). Is this series Convergent or Divergent?

Solution: Let \( f(x) = \frac{1}{x^2 + 1} \) for all \( x \geq 1 \).

Then, \( f(x) \) is positive and continuous on \([1, \infty)\) and \( f(n) = a_n \) for all \( n \geq 1 \).

Now, \( f'(x) = \frac{d}{dx} \left( (x^2 + 1)^{-1} \right) = (-1)(x^2 + 1)^{-2}(2x) = \frac{-2x}{(x^2 + 1)^2} < 0 \), for all \( x \geq 1 \).

Since \( f'(x) < 0 \) on \([1, \infty)\), \( f(x) \) is decreasing on \([1, \infty)\).

So, the Integral Test Applies.

\[
\int_{1}^{\infty} \frac{1}{x^2 + 1} \, dx = \lim_{t \to \infty} \left( \int_{1}^{t} \frac{1}{x^2 + 1} \, dx \right) = \lim_{t \to \infty} \left[ \left( \tan^{-1}(x) \right) \right]_{1}^{t} \\
- \lim_{t \to \infty} \left( \tan^{-1}(t) - \frac{\pi}{4} \right) = \left( \frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{4}.
\]

So, \( \int_{1}^{\infty} \frac{1}{x^2 + 1} \, dx \) is Convergent.

Since the integral \( \int_{1}^{\infty} \frac{1}{x^2 + 1} \, dx \) is Convergent,

the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \) is Convergent by the Integral Test.

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