Applying the Integral Test Correctly and the Required Sentences for the Integral Test:

The Integral Test applies to a series $\sum_{n=1}^{\infty} a_n$ such that there exists a function $y = f(x)$ which satisfies

the **FOUR SPECIAL CONDITIONS :**

(1) Function f is continuous on $[1, \infty)$ (or on $[K, \infty)$ for some $K \geq 1$),

(2) Function f is positive (that is, $f(x) \geq 0$) on $[1, \infty)$ (or on $[K, \infty)$ for some $K \geq 1$),

(3) $f(n) = a_n$ for all $n \geq 1$ (for all $n \geq K$ for some $K \geq 1$),

(4) Function f is **decreasing** on $[1, \infty)$ (or on $[K, \infty)$ for some $K \geq 1$).

The FIRST TASK for BOTH CASES, the Case of Convergence and the Case of Divergence:

The FIRST TASK in applying the integral test is to STATE that Special Conditions (1), (2), and (3) are true about Function f, and also to STATE and to VERIFY that Special Condition (4) is true about Function f.

For Special Conditions (1), (2), and (3), it is enough only to state "Function f is continuous and positive on $[1, \infty)$ and $f(n) = a_n$ for all $n \geq 1$."

The verification that Special Condition (4) is true about Function f, that is, that Function f is decreasing on $[1, \infty)$, can be accomplished algebraically by proving that, whenever $1 \leq x_1 \leq x_2$, $f(x_1) \geq f(x_2)$.

However, Special Condition (4) is frequently verified by showing that the derivative f' is negative

(that is, $f'(x) < 0$) on $[1, \infty)$ (or on $[K, \infty)$ for some $K \geq 1$).

Sentences and Explanations concerning the Special Conditions (1), (2), (3), and (4) are REQUIRED.

After these sentences and explanations have been provided, you must then state:

"The Integral Test Applies."

Then, after stating "The Integral Test Applies," you must **evaluate the correct improper integral**

$$\int_{1}^{\infty} f(x)\,dx \; .$$

The Final Task in Applying the Integral Test: Writing the Required Sentence.

CASE 1: The improper integral $\int_1^\infty f(x)\,dx$ **is CONVERGENT:**

In this case, you must write this **Required Sentence**:

"Since the integral $\int_1^\infty f(x)\,dx$ is Convergent, the series $\sum_{n=1}^{\infty} a_n$ is Convergent **by the Integral Test.**"

CASE 2: The improper integral $\int_1^\infty f(x)\,dx$ **is** DIVERGENT**:**

In this case, you must write this **Required Sentence**:

"Since the integral $\int_1^\infty f(x)\,dx$ is Divergent, the series $\sum_{n=1}^{\infty} a_n$ is Divergent **by the Integral Test.**"

On the next page is an example of a complete and correct presentation of an application of the Integral Test.

FOR EXAMPLE: Consider the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$. **Is this series Convergent or Divergent?**

Solution: Let $f(x) = \frac{1}{x^2 + 1}$ for all $x \geq 1$.

Then, $f(x)$ is positive and continuous on $[1, \infty)$ and $f(n) = a_n$ for all $n \geq 1$.

Now, $f'(x) = \frac{d}{dx}\left((x^2 + 1)^{-1} \right) = (-1)(x^2 + 1)^{-2}(2x) = \frac{-2x}{x^2 + 1} < 0$, for all $x \geq 1$.

Since $f'(x) < 0$ on $[1, \infty)$, $f(x)$ is decreasing on $[1, \infty)$.

So, **the Integral Test Applies.**

$$\int_1^{\infty} \frac{1}{x^2 + 1}\,dx = \lim_{t \to \infty} \left(\int_1^{t} \frac{1}{x^2 + 1}\,dx \right) = \lim_{t \to \infty} \left[\left(\tan^{-1}(x) \right) \right]_1^t$$

$$= \lim_{t \to \infty} \left(\tan^{-1}(t) - \frac{\pi}{4} \right) = \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{4} .$$

So, $\int_1^{\infty} \frac{1}{x^2 + 1}\,dx$ is Convergent.

Since the integral $\int_1^{\infty} \frac{1}{x^2 + 1}\,dx$ is Convergent,

the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ is Convergent by the Integral Test.

REQUIRED SENTENCE