Approximating summation s with s_n

When the series $s = \sum_{n=1}^{\infty} a_n$ is convergent,

there is usually no formula to figure out s exactly.

$$s = \underbrace{a_1 + a_2 + a_3 + \cdots + a_n}_{s_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{R_n = \text{The Remainder Sum}}$$

$s_n + R_n = s$, so

$$\underline{R_n = s - s_n}.$$

$s - s_n = R_n =$ The Error we have when we "use s_n as an approximation of s"

Thus, $R_n = \sum_{k=n+1}^{\infty} a_k = s - s_n$ is the error in $s_n \approx s$.

We say we are "ESTIMATING THE LEVEL OF ERROR" when we find a small number t such that we are certain that

$$|\text{The ERROR}| \leq t.$$

FACT:

When $s = \sum_{n=1}^{\infty} a_n$ has $a_n \geq 0$ for all n

and the series is convergent and has been shown to be convergent by the Integral Test with a function $f(x)$, satisfying conditions ①, ②, ③, ④ stated in the Integral Test,

Then, for each n, (As always)

the ERROR in $S_n \approx S$ is R_n, $R_n \geq 0$

and $R_n \leq \int_n^{\infty} f(x)\,dx$, so

$t = \int_n^{\infty} f(x)\,dx$ IS A LEVEL OF ERROR in using $S_n \approx S$.

A level of ERROR IN THIS CASE

So. $\boxed{R_n = s - s_n = |\text{ERROR}| \leq \int_n^{\infty} f(x)\,dx}$

Problem: Estimate A LEVEL OF ERROR in using $S_{10} \approx S = \sum_{n=1}^{\infty} \frac{1}{n^4}$.

Sol'n THE ERROR in "$S_{10} \approx S$" $= R_{10} = \sum_{k=11}^{\infty} a_k$.

By the p-series test, $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent to s, and $f(x) = \frac{1}{x^4}$ is a function satisfying conditions ①, ②, ③ and ④ of the Integral Test.

Therefore, by the FACT above,

$R_{10} = s - s_{10}$ = The ERROR in $s_{10} \approx s$ has the property that $R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4}\,dx$

Integrating, we get $\frac{x^{-3}}{-3}$

$$\int_{10}^{\infty} \frac{1}{x^4} = \lim_{t\to\infty} \int_{10}^{t} x^{-4}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3x^3}\right]_{10}^{t}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{3t^3} - \left(-\frac{1}{3 \times 1000}\right)\right] = \frac{1}{3000} = \frac{1}{3(10)^3}$$

Thus $\underset{n=10}{R_n}$ = ERROR in $s_{10} \approx s \leq \int_{10}^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3000}$.

$\therefore$ ERROR $\underset{n=10}{R_n} \leq 0.000333\ldots$ = The Level OF Error.

THUS, when sum $s = \sum_{n=1}^{\infty} \frac{1}{n^4}$, the LEVEL OF ERROR in $s_{10} \approx s$ is $\frac{1}{3(10)^3} = 0.000333\ldots$

RELATED PROBLEM:

When using $S_n \approx S = \sum_{n=1}^{\infty} \frac{1}{n^4}$,

for which n will the ERROR in $S_n \approx S$

be less than or equal to 0.0000001 ?

Sol'n: The ERROR in $S_n \approx S$ is R_n,

and, since the Integral Test Applies with

$f(x) = \frac{1}{x^4}$, $\quad$ ERROR $= R_n \leq \int_n^{\infty} \frac{1}{x^4}\,dx$

and $\int_n^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_n^t x^{-4}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3x^3}\right]_n^t$

$$= \lim_{t\to\infty} \left(-\frac{1}{3t^3} - \left(-\frac{1}{3n^3}\right)\right) = \frac{1}{3n^3}.$$

So, $R_n =$ ERROR $\leq \frac{1}{3n^3}$. So, We need:

$\frac{1}{3n^3} \leq 0.0000001 \Rightarrow 1 \leq 0.0000001 * 3 * n^3$

$\Rightarrow \frac{1}{0.0000003} \leq n^3 \Rightarrow \sqrt[3]{\frac{1}{0.0000003}} \leq n$

That is, $n \geq 149.38$.

Thus, for $n = 150$ (or greater), the error R_n

in $S_n \approx S$ is such that $|\text{ERROR}| \leq 0.0000001$. ∎