Tests for the Convergence or Divergence of Infinite Series

Three useful sequence limits:

$$\begin{cases} \text{If } |x| < 1, \text{ then } x^n \to 0 \text{ as } n \to \infty \\ \text{If } x = 1, \text{ then } 1^n \to 1 \text{ as } n \to \infty \\ \text{If } x > 1, \text{ then } x^n \to \infty \text{ as } n \to \infty \end{cases}$$

Series that we know Converge:

A. For any fixed $p > 1$: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ **converges. (These type series are called p-series.)**

Examples: With $p = 2$, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ **converges.**

With $p = 3/2$, $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ **converges.**

B. The Geometric Series when $|r| < 1$ will converge to $1/(1-r)$.

In this case: $1 + r + r^2 + r^3 + \cdots = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

Series that we know Diverge:

A) The Harmonic Series: $\sum_{k=1}^{\infty} \frac{1}{k} \to \infty$ **B) The p-series when $p \le 1$:** $\sum_{k=1}^{\infty} \frac{1}{k^p} \to \infty$

C) The Geometric Series when $|r| \ge 1$ will diverge.

In this case: $1 + r + r^2 + r^3 + \cdots = \sum_{k=0}^{\infty} r^k$ *diverges and if* $r \ge 1$, $\sum_{k=0}^{\infty} r^k \to \infty$

The Major Tests for Convergence and Divergence for Infinite Series

1) The Basic Divergence Test: *If* $a_k \xrightarrow{\text{Does NOT}} 0$, *then* $\sum_{k=0}^{\infty} a_k$ *diverges*.

2) The Integral Test: If f is continuous, decreasing, and positive on $[1, \infty)$,

then with $a_k = f(k)$:

$$\begin{cases} \sum a_k = \sum f(k) \text{ Converges, if } \int_1^{\infty} f(x)\,dx \text{ Converges .} \\ \sum a_k = \sum f(k) \text{ Diverges, if } \int_1^{\infty} f(x)\,dx \text{ Diverges .} \end{cases}$$

3) The Basic (Direct) Comparison Test: Let $\sum a_k$ and $\sum b_k$ be series with

all non-negative terms such that for ALL BIG k, $a_k \leq b_k$, then:

$$\begin{cases} \text{If } \sum b_k \text{ converges, then } \sum a_k \text{ converges, too.} \\ \text{If } \sum a_k \text{ diverges, then } \sum b_k \text{ diverges, too.} \end{cases}$$

4) The Limit Comparison Test: Let $\sum a_k$ be a series with all non-negative terms and let $\sum b_k$ be series with all positive terms.

If the sequence $\frac{a_k}{b_k} \to L > 0$, *where L is a real number*, then

either both of the series converge or both of the series diverge .

5) The Root Test:

Let $\sum a_k$ be a series with all non-negative terms such that

$$a_k^{1/k} \to p \quad as \quad k \to \infty .$$

- a) If $p < 1$, then the series $\sum a_k$ converges.
- b) If $p > 1$, then the series $\sum a_k$ diverges.
- c) If $p = 1$, then the results of the test are inconclusive.

6) The Ratio Test:

Let $\sum a_k$ be a series with all positive terms such that

$$\frac{a_{k+1}}{a_k} \to L \quad as \quad k \to \infty .$$

- a) If $L < 1$, then the series $\sum a_k$ converges.
- b) If $L > 1$, then the series $\sum a_k$ diverges.
- c) If $L = 1$, then the results of the test are inconclusive.