

MODEL SOLUTIONS FOR TWO PROBLEMS,

(1) AN Integral Test application and

(2) #36 (a, b) from Section 11.3.

Problem 1: $\left\{ \text{Is } \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \text{ ConD?} \right\}$

#22 Consider the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} = \sum_{n=2}^{\infty} a_n$.

The formula for the n^{th} term of $\{a_n\}_{n=2}^{\infty}$ is

$$a_n = \frac{1}{n(\ln n)^2} \quad \left| \quad \text{The function } f(x) = \frac{1}{x(\ln x)^2}, \right.$$

$$x \geq 2,$$

is the function to use when performing the integral test (if it applies).

SHOWING THAT The Integral Test Applies:

① $f(x)$ is continuous on $[2, \infty)$ since the only values of x for which $f(x)$ is not defined are $x=0$ and $x=1$.

② $f(x) > 0$ on $[2, \infty)$ since $f(x) = \frac{1}{(\text{positive})(\text{number})^2} > 0$.

③ $f(n) = \frac{1}{n(\ln n)^2} = a_n$ when $x=n$, $n=2, 3, 4, \dots$

$$\begin{aligned} \text{④ } f(x) &= (x^{-1})(\ln x)^{-2} \\ f'(x) &= (-1)(x^{-2})(\ln x)^{-2} + (x^{-1})(-2)(\ln x)^{-3}\left(\frac{1}{x}\right) \\ &= \frac{-1}{x^2(\ln x)^2} + \frac{-2}{x^2(\ln x)^3} < 0, \text{ so } f \text{ is decreasing!} \end{aligned}$$

$\therefore$ THE INTEGRAL TEST APPLIES!!

Sec 11.3, #22 (continued)

If $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$ converges, then $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Converges.

If $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$ diverges, then $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ diverges.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t (\ln x)^{-2} \left(\frac{1}{x}\right) dx \quad \left[\begin{array}{l} \text{let} \\ u = \ln x \\ du = \frac{1}{x} dx \end{array} \right]$$

$$= \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} u^{-2} du = \lim_{t \rightarrow \infty} \left(-\frac{1}{u} \right) \Big|_{\ln 2}^{\ln t}$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{\ln t} - \frac{-1}{\ln 2} \right) = +\frac{1}{\ln 2}$$

Since $-\frac{1}{\ln t} \rightarrow 0$ as $t \rightarrow \infty$,

Then $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$ is convergent and it converges to $\frac{1}{\ln 2}$.

By the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is

convergent because the improper integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx \text{ is Convergent.}$$

Sec 11.3, (a) and (d)

As you read through this solution, have the handout "The Remainder R_n " nearby for a summary of the basic principles involved.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} a_n$.

The first issue to consider is C or D?

This series is a p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 4$.

The p -series converges when $p > 1$. Now, $4 > 1$.

$\therefore$ The series converges; say it converges to S .

$$S_0, \sum_{n=1}^{\infty} \frac{1}{n^4} = S.$$

(a) Consider S_{10} as an approximation to the series' sum S ; that is, $S_{10} \approx S$.

Question: ① What # is S_{10} ?

② Estimate the ^{LEVEL} OF ERROR involved when using S_{10} as an approximation to S .

Question ①: $S_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots + \frac{1}{10^4}$

Using a calculator, $S_{10} \approx 1.082037$.
(and rounding to 6 decimal places).

Sec 11.3 #36 (a) (Continued)

(4)

a)

Question (2): Estimate ^{the} _{LEVEL} _{OF ERROR} in $S_{10} \approx S$.

For any S_n , when we consider S_n as an approximation to S , $S_n \approx S$, the error in this approximation is exactly $R_n = S - S_n$.

$$\text{This error } R_n = S - S_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{10} a_n = \sum_{n=11}^{\infty} a_n.$$

Thus, the error $R_n = S - S_n$ is less than or equal to...?

$$R_n \leq \int_n^{\infty} \frac{1}{x^4} dx$$

$$(*) \text{ In fact, } \int_{n+1}^{\infty} \frac{1}{x^4} dx \leq R_n \leq \int_n^{\infty} \frac{1}{x^4} dx$$

$\leftarrow R_n = S - S_n$

We are asked here in this part (a) to estimate the maximum possible error in the use of S_{10} as an approximation of S , $S_{10} \approx S$.

So, we only need the second inequality here.

$$\text{Error} = S - S_{10} \approx R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4} dx = \lim_{t \rightarrow \infty} \int_{10}^t \frac{1}{x^4} dx = \dots = \frac{1}{3(10)^3}$$

$$\text{Thus, Error } R_{10} \leq \frac{1}{3000} = 0.000333\dots$$

Sec 11.3, #36 (Continued)

(5)

(d) Find a value of n so that s_n is within 0.00001 of the series' sum S .

That is, find a value for n so that the error

$$R_n = S - s_n \leq 0.00001.$$

We know from line (*) a few pages back that

$$R_n \leq \int_n^{\infty} \frac{1}{x^4} dx = \frac{1}{3n^3}.$$

We can be sure that $R_n \leq 0.00001$ if we choose n so that

$$\int_n^{\infty} \frac{1}{x^4} dx = \frac{1}{3n^3} \leq 0.00001$$

So we solve the inequality $\frac{1}{3n^3} \leq 0.00001$.
[multiplying by $3n^3$]

$$1 \leq (0.00001)(3n^3)$$

[Dividing by $(0.00001)(3)$]

$$\frac{1}{(0.00001)(3)} \leq n^3$$

[Taking Cubed Roots $y^{1/3}$]

$$\sqrt[3]{\frac{1}{(0.00003)}} \leq n$$

When $n > \sqrt[3]{1/0.00003} \approx 32.18$, Error $S - s_n = R_n \leq 0.00001$.

When $n \geq 33$, $R_n = \text{Error in } S_n \text{ \& } S \text{ is } \leq 0.00001$.

Sec 11.3 , #36 (continued)

(6)

Bonus Question: Find the sum of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \text{correct to} \\ \text{three decimal places}$$

"Correct to 3 decimal places" means

$$\text{ERROR} \leq 0.0005$$

3 zeros

"Correct to 4 decimal places"

$$\text{means ERROR} \leq 0.00005$$

4 zeros

etc. [HERE, we seek correctness to 3 places:

$$\text{The Error in } S_n \approx S \text{ is } R_n \text{ and } R_n \leq \int_n^{\infty} \frac{1}{x^4} dx = \frac{1}{3n^3}$$

$$\text{Solve } \frac{1}{3n^3} \leq 0.0005$$

$$\frac{1}{(0.0005)(3)} \leq n^3 \quad \begin{aligned} &1 \leq (0.0005)(3n^3) \\ &\rightarrow n \geq \sqrt[3]{\frac{1}{0.0015}} \end{aligned}$$

$$n \geq 8.736 \quad \therefore n=9 \text{ will do}$$

$S_9 \approx S$ is CORRECT TO 3 places

$$S = 1.082, \text{ correct to 3 places.}$$

Sec. 11.3 #36 (continued)

[Read this only if (7)
you are interested.
This is not covered in class.]

(b) It says: "Use (3) with $n=10$ to give an improved estimate of the sum S "

(That is, an improvement over S_{10} as an estimate of the series' sum S .)

When it says "use (3)", it is saying to use the bounds for the sum S coming from line (*) on the previous page.

Since $R_n = S - S_n$, we can get an inequality for S alone by adding S_n to both (all three, really) sides of the inequalities in (*).

$$S_n + \int_{n+1}^{\infty} \frac{1}{x^4} dx \leq S \leq S_n + \int_n^{\infty} \frac{1}{x^4} dx$$

Here, with $n=10$:

$$S_{10} + \int_{11}^{\infty} \frac{1}{x^4} dx \leq S \leq S_{10} + \int_{10}^{\infty} \frac{1}{x^4} dx$$

We can work to get a formula for $\int_k^{\infty} \frac{1}{x^4} dx$:

$$\int_k^{\infty} \frac{1}{x^4} dx = \lim_{t \rightarrow \infty} \int_k^t \frac{1}{x^4} dx = \dots = \frac{1}{3k^3}$$

$$\text{Thus } \int_{11}^{\infty} \frac{1}{x^4} dx = \frac{1}{3(11)^3} \text{ and } \int_{10}^{\infty} \frac{1}{x^4} dx = \frac{1}{3(10)^3}$$

Sec 11.3, #36

(8)

(b) (continued)

$$\text{So, } \underbrace{S_{10} + \frac{1}{3(11^3)}}_{\text{call this } \underline{a}} \leq S \leq \underbrace{S_{10} + \frac{1}{3(10^3)}}_{\text{call this number } \underline{b}}$$

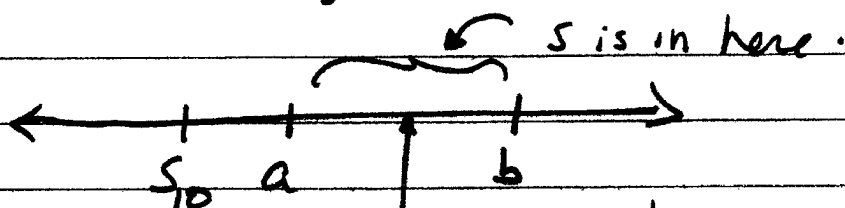
In particular:

$$1.082037 + 0.000250 = 1.082287 \leq S$$

$$\text{and } S \leq 1.082370 = 1.082037 + 0.000333$$

$$\text{That is, } \underbrace{1.082287}_{=\underline{a}} \leq S \leq \underbrace{1.082370}_{=\underline{b}}$$

On a number line, this situation is like this:



Midpoint =
Approximation
to S

$$\frac{a+b}{2}$$

The midpoint between a and b is
an improved approximation to S
compared with S_{10} as an approxi-
mation to S .

$$\therefore \text{The Improved Approximation to } S \text{ is } \frac{a+b}{2} = \frac{1.082287 + 1.082370}{2}$$

$$= \frac{2.164657}{2} = \underline{\underline{1.08233}} \text{ when Rounded to 5 places.}$$