Re-arranging the Terms of a Conditionally Convergent Series

Facts: If $\sum_{n=1}^{\infty} a_n$ is Absolutely Convergent,

then every series that results from re-arranging the terms is also absolutely convergent and it converges to the same sum s.

If $\sum_{n=1}^{\infty} a_n$ is Conditionally Convergent only,

then re-arranging the terms can result in a series that converges to a different sum. In fact, the terms of the conditionally convergent can be re-arranged to produce a series that converges to any desired sum s , or even a series that is divergent.

Example:

Suppose $$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} + \cdots$$

This alternating series (alternating eventually at least) is seen to be convergent by the Alternating Series Test.

However, its absolute value series is

$$\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \frac{1}{5} + \cdots$$

and it diverges by comparison with the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$ **which is divergent.**

Therefore, $$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} + \cdots$$ **is Conditionally Convergent.**

This implies that the terms of the series can be re-arranged so that the resulting series has a sequence of partial sums which converges to any desired sum s , or so that the resulting series even diverges!

Such a re-arrangement of terms resulting in a divergent series will be illustrated here.

Before we begin, note that, because the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

is divergent and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ **, all "tails" of this series are also divergent,**

that is, for any positive integer N ,

$$\sum_{n=N}^{\infty} \frac{1}{n} = \frac{1}{N} + \frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} + \cdots = \infty .$$

For example,

$$\sum_{n=1000}^{\infty} \frac{1}{n} = \frac{1}{1000} + \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \cdots = \infty ,$$

or else (by adding 999 terms) we would be able to show that the harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{999} + \frac{1}{1000} + \frac{1}{1001} + \cdots$ **is convergent,**

which it is not.

So, no matter where we start, say at $\frac{1}{N}$ **, we can add together a BLOCK B of terms from the "tail",**

say ending at $\frac{1}{M}$ **, which add up to a sum which is greater than or equal to 1 . That is, there**

is some number M > N so that

$$\sum \text{BLOCK B} = \frac{1}{N} + \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{M} \geq 1 .$$

Now, consider again the original conditionally convergent series

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} + \cdots .$$

We can group together the terms so that we see that the series converges to the sum s = 1 :

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} + \cdots$$

$$= 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{5}\right) + \cdots = 1$$

With this arrangement of the terms a_n, the sequence of partial sums is the following:

$$s_1, \quad s_2, \quad s_3, \quad s_4, \quad s_5, \quad s_6, \quad s_7, \quad s_8, \quad s_9, \ \ldots \quad =$$

$$1, \quad 1\tfrac{1}{2}, \quad 1, \quad 1\tfrac{1}{3}, \quad 1, \quad 1\tfrac{1}{4}, \quad 1, \quad 1\tfrac{1}{5}, \quad 1, \cdots \quad \to 1$$

So, $s = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = 1$.

We will now describe a re-arrangement of the terms a_n so that the series that results is divergent. First, let's look at the resources available for the re-arrangement:

$$1$$

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \frac{1}{7}, \quad \frac{1}{8}, \quad \ldots$$

$$-\frac{1}{2}, \quad -\frac{1}{3}, \quad -\frac{1}{4}, \quad -\frac{1}{5}, \quad -\frac{1}{6}, \quad -\frac{1}{7}, \quad -\frac{1}{8}, \quad \ldots$$

The terms in the middle row are the terms of the Harmonic Series (minus the first term) which form a divergent series when added together. This means that we can split this row into BLOCKS of terms

$B_1, B_2, B_3, B_4, \ldots$, **each of which has a sum ≥ 1 .**

For example, $\sum$ BLOCK B_1 can be $\sum B_1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$. **Let $n_1 = 4$.**

So, the terms in the second row can be split up into such blocks:

$$\underbrace{\frac{1}{2},\quad \frac{1}{3},\quad \frac{1}{4},}_{\mathbf{B_1}}\qquad \underbrace{\frac{1}{5},\quad \frac{1}{6},\quad \ldots,\quad \frac{1}{n_2},}_{\mathbf{B_2}}\qquad \underbrace{\frac{1}{n_2+1},\quad \ldots,\quad \frac{1}{n_3},}_{\mathbf{B_3}}\qquad \ldots$$

We can then re-arrange the terms a_n of the of the original series $\sum_{n=1}^{\infty} a_n$ to produce the series

$$\sum_{n=1}^{\infty} c_n = 1 + \left(\sum BLOCK\ B_1\right) - \frac{1}{2} \qquad \left(\leftarrow sum \geq 1 - \frac{1}{2} \geq \frac{1}{2}\right)$$

$$+ \left(\sum BLOCK\ B_2\right) - \frac{1}{3} \qquad \left(\leftarrow sum \geq 1 - \frac{1}{3} \geq \frac{1}{2}\right)$$

$$+ \left(\sum BLOCK\ B_3\right) - \frac{1}{4} \qquad \left(\leftarrow sum \geq 1 - \frac{1}{4} \geq \frac{1}{2}\right)$$

$$+ \cdots \qquad \textit{etc.}$$

Therefore, $$\sum_{n=1}^{\infty} c_n \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty .$$

Therefore, $\sum_{n=1}^{\infty} c_n$,

a re-arrangement of the terms of the conditionally convergent series $\sum_{n=1}^{\infty} a_n$,

DIVERGES!