MODEL SOLUTIONS FOR TWO PROBLEMS,

(1) AN Integral Test application and
(2) #36 (a, b) from Section 11.3.

Problem 1: Is $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ C or D?

#22 Consider the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} = \sum_{n=2}^{\infty} a_n$.

The formula for the n^{th} term of $\{a_n\}_{n=2}^{\infty}$ is

$\boxed{a_n = \frac{1}{n(\ln n)^2}.}$ The function $f(x) = \frac{1}{x(\ln x)^2}$, $x \geq 2$,

is the function to use when performing the integral test (if it applies).

SHOWING THAT The Integral Test Applies:

① $f(x)$ is continuous on $[2, \infty)$ since the only values of x for which $f(x)$ is not defined are $x = 0$ and $x = 1$.

② $f(x) > 0$ on $[2, \infty)$ since
$$f(x) = \frac{1}{(\text{positive})(\text{number})^2} > 0.$$

③ $f(n) = \frac{1}{n(\ln n)^2} = a_n$ when $x = n$, $n = 2, 3, 4, \ldots$

④ $f(x) = (x^{-1})((\ln x)^{-2})$
$$f'(x) = (-1)(x^{-2})(\ln x)^{-2} + (x^{-1})\left(-2(\ln x)^{-3}\left(\tfrac{1}{x}\right)\right)$$
$$= \frac{-1}{x^2(\ln x)^2} + \frac{-2}{x(\ln x)^3} < 0,$$ so f is decreasing!

$\therefore$ THE INTEGRAL TEST APPLIES!!

Sec 11.3, #22 (continued)

If $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges, then $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Converges.

If $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ diverges, then $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ diverges.

$$\int_2^\infty \frac{1}{x(\ln x)^2}dx = \lim_{t\to\infty}\int_2^t (\ln x)^{-2}\left(\tfrac{1}{x}\right)dx$$

$\boxed{\begin{array}{l}\text{let} \\ u = \ln x \\ du = \frac{1}{x}dx\end{array}}$

$$= \lim_{t\to\infty}\int_{\ln 2}^{\ln t} u^{-2}\,du = \lim_{t\to\infty}\left(\left(-\frac{1}{u}\right)\Big|_{\ln 2}^{\ln t}\right)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{\ln t} - \frac{-1}{\ln 2}\right) = +\frac{1}{\ln 2}$$

Since $\frac{-1}{\ln t} \to 0$ as $t \to \infty$.

Thus $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ is convergent and it converges to $\frac{1}{\ln 2}$.

By the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent because the improper integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ is Convergent.

Sec 11.3, (a) and (d)

As you read through this solution, have the handout "The Remainder R_n" nearby for a summary of the basic principles involved.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} a_n$.

The first issue to consider is C or D?

This series is a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 4$.

The p-series converges when $p > 1$. Now, $4 > 1$.

$\therefore$ The series converges; say it converges to s.

So, $\sum_{n=1}^{\infty} \frac{1}{n^4} = s$.

(a) Consider s_{10} as an approximation to the series' sum s; That is, $s_{10} \approx s$.

Questions: ① What # is s_{10}?

② Estimate the LEVEL OF ERROR involved when using s_{10} as an approximation to s.

Question ①: $s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4}$

Using a calculator, $s_{10} \approx 1.082037$.
(and rounding to 6 decimal places).

Sec 11.3, #36 (a) (Continued)

a)

Question ②: Estimate the LEVEL OF ERROR in $S_{10} \approx S$.

For any S_n, when we consider S_n as an approximation to S, $S_n \approx S$, the error in this approximation is exactly $R_n = S - S_n$.

This error $R_n = S - S_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{10} a_n = \sum_{n=11}^{\infty} a_n$.

Thus, the error $R_n = S - S_n$ is less than or equal to ?

$$R_n \leq \int_n^{\infty} \frac{1}{x^4}\, dx$$

(*) In fact, $\displaystyle\int_{n+1}^{\infty} \frac{1}{x^4}\, dx \leq \underbrace{R_n}_{= S - S_n} \leq \int_n^{\infty} \frac{1}{x^4}\, dx$

We are asked here in this part (a) to estimate the maximum possible error in the use of S_{10} as an approximation of S, $S_{10} \approx S$.

So, we only need the second inequality here.

$$\text{ERROR} = S - S_{10} = R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4}\, dx = \lim_{t \to \infty} \int_{10}^{t} \frac{1}{x^4}\, dx = \cdots = \frac{1}{3(10)^3}$$

$$\boxed{\text{Thus, ERROR } R_{10} \leq \frac{1}{3000} = 0.000333\ldots}$$

Sec 11.3, #36 (Continued)

(d) Find a value of n so that s_n is within 0.00001 of the series' sum s.

That is, find a value for n so that the error

$$R_n = s - s_n \leq 0.00001.$$

We know from line (*) a few pages back that

$$R_n \leq \int_n^\infty \frac{1}{x^4}\,dx = \frac{1}{3n^3}.$$

We can be sure that $R_n \leq 0.00001$ if we choose n so that

$$\int_n^\infty \frac{1}{x^4}\,dx = \frac{1}{3n^3} \leq 0.00001$$

So, we solve the inequality $\frac{1}{3n^3} \leq 0.00001$.

[multiplying by $3n^3$]

$$1 \leq (0.00001)(3n^3)$$

[Dividing by $(0.00001)(3)$]

$$\frac{1}{(0.00001)(3)} \leq n^3.$$

[Taking Cubed Root $y^{1/3}$]

$$\sqrt[3]{\frac{1}{(0.00003)}} \leq n$$

$$\boxed{\begin{array}{l} \text{When } n > \sqrt[3]{1/0.00003} \approx 32.18, \text{ Error } s - s_n = R_n \leq 0.00001 \\ \text{When } n \geq 33, \ R_n = \text{ERROR in } s_n \text{ as } s \text{ is } \leq 0.00001. \end{array}}$$

Sec 11.3, #36 (Continued)

Bonus Question: Find the sum of the series $s = \sum_{n=1}^{\infty} \frac{1}{n^4}$ correct to three decimal places.

"Correct to 3 decimal places" means ERROR ≤ 0.0005.
$\underbrace{000}$ ← 3 zeros

"Correct to 4 decimal places" means ERROR ≤ 0.00005
$\underbrace{0000}$ ← 4 zeros

⋮
etc.

HERE, we seek correctness to 3 places:

The Error in $S_n \approx s$ is R_n and $R_n \le \int_n^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3n^3}$

Solve $\frac{1}{3n^3} \le 0.0005$.

$1 \le (0.0005)(3n^3)$

$\frac{1}{(0.0005)(3)} \le n^3$ → $n \ge \sqrt[3]{\frac{1}{0.0015}}$

$n \geq 8.736$ ∴ $n = 9$ will do

$S_9 \approx s$ is CORRECT TO 3 places

$s = 1.082$, correct to 3 places.

Sec. 11.3 # 36 (continued)

[Read this only if you are interested. This is not covered in class.

(b) It says: "Use (3) with $n=10$ to give an improved estimate of the sum s"
(That is, an improvement over s_{10} as an estimate of the series' sum s.)

When it says "use (3)", it is saying to use the bounds for the sum s coming from line (*) on the previous page.

Since $R_n = s - s_n$, we can get an inequality for s alone by adding s_n to both (all three, really) sides of the inequalities in (*).

$$s_n + \int_{n+1}^{\infty} \frac{1}{x^4}\,dx \le s \le s_n + \int_n^{\infty} \frac{1}{x^4}\,dx$$

Here, with $n=10$:

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^4}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^4}\,dx$$

We can work to get a formula for $\int_k^{\infty} \frac{1}{x^4}\,dx$:

$$\int_k^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_k^t \frac{1}{x^4}\,dx = \cdots = \frac{1}{3k^3}$$

Thus $\int_{11}^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3(11)^3}$ and $\int_{10}^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3(10)^3}$

Sec 11.3, #36

(b) (continued)

So, $\underbrace{S_{10} + \frac{1}{3(11^3)}}_{\text{call this \# } a} \leq S \leq \underbrace{S_{10} + \frac{1}{3(10^3)}}_{\text{call this number } b}$.

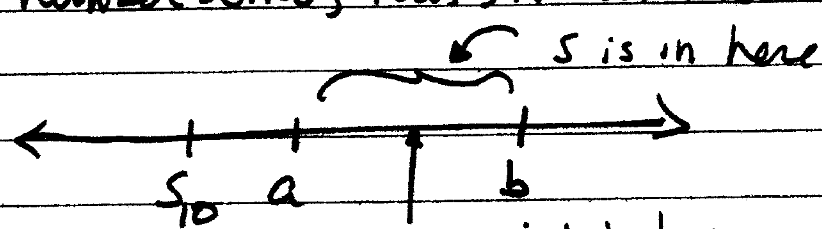
In particular:

$1.082037 + 0.000250 = 1.082287 \leq S$

and $S \leq 1.082370 = 1.082037 + 0.000333$

That is, $\underbrace{1.082287}_{=a} \leq S \leq \underbrace{1.082370}_{=b}$

On a number line, this situation is like this:



The midpoint between a and b is an improved approximation to S compared with S_{10} as an approximation to S.

Midpoint Approximation to S $= \frac{a+b}{2}$

$\therefore$ The Improved Approximation to S is $\frac{a+b}{2} = \frac{1.082287 + 1.082370}{2}$

$= \frac{2.164657}{2} = \underline{1.08233}$ when Rounded to 5 places.