SPECIAL NOTES ON INFINITE SERIES

When talking about the series $\sum_{k=1}^{\infty} a_k$, there are two different sequences being discussed.

One is the Sequence of Terms of the series = $\{ a_k \}$ and

the other is the Sequence of Partial Sums = $\{ s_n \}$ where $s_n = \sum_{k=1}^{n} a_k$.

<u>Explanations:</u>

$\{ a_k \}$: $a_1, a_2, a_3, a_4, \ldots$ is the Sequence of Terms of the series.

Thus, for the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots,$

the sequence of terms is $\{ a_k \} = \{ 1/k \} : 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots$

Thus, the sequence of terms, $\{ 1/k \} : 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots \rightarrow 0$ **as** $k \rightarrow \infty$

The series $\sum_{k=1}^{\infty} \frac{1}{k}$ **is not converging to 0, but the sequence of its terms is converging to 0.**

For the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\ldots,$

the sequence of terms is $\{ a_k \} = \{ 1/(k^2) \} : 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\ldots$

Thus, the sequence of terms, $\{ 1/(k^2) \} : 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\ldots \rightarrow 0$ **as** $k \rightarrow \infty$

The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ **converges (but not to 0) ; however, the sequence of its terms *is* converging to 0.**

THE SEQUENCE OF PARTIAL SUMS

$\{ s_n \} : s_1, s_2, s_3, s_4, \ldots$ **is the Sequence of Partial Sums of the series.**

Thus, for the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots ,$

$s_1 = 1 , \quad s_2 = 1 + \frac{1}{2} , \quad s_3 = 1 + \frac{1}{2} + \frac{1}{3} , \quad s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} ,$ **and so on ...**

In fact, $s_1 = 1, \quad s_2 = 3/2 , \quad s_3 = 11/6, \quad s_4 = 25/12 , \quad \ldots\ldots \rightarrow \infty .$

Whenever we say "The series $\sum_{k=1}^{\infty} a_k$ **converges" , what we really mean is**

"The sequence of partial sums $\{ s_n \} \rightarrow L$ **for some limit L "**

Whenever we say "The series $\sum_{k=1}^{\infty} a_k$ **diverges", what we really mean is**

"The sequence of partial sums $\{ s_n \}$ **diverges and does not have a limit."**

When we say the series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ **diverges or** $\sum_{k=1}^{\infty} \frac{1}{k} \rightarrow \infty ,$

what we really mean is " The sequence of partial sums $\{ s_n \}$ **diverges." or**

" The sequence of partial sums $\{ s_n \} \rightarrow \infty$ **. (Note that the sequence of terms** $\frac{1}{k} \rightarrow 0$ **.)**

When we say the series $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\ldots$ **converges or** $\sum_{k=1}^{\infty} \frac{1}{k^2} \rightarrow$ **limit L ,**

what we really mean is " The sequence of partial sums $\{ s_n \}$ **converges." or**

" The sequence of partial sums $\{ s_n \} \rightarrow$ **limit L** > 0 **. (Note that the sequence of terms** $\frac{1}{k^2} \rightarrow 0$ **.)**

Example Exercises and Their Solutions:

Exercise #1: Consider the series $\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = \frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \frac{3}{(3+1)!} + \cdots$.

a) Write down the first three entries in the Sequence of Terms $\{ a_k \}$ of this series.

b) Write down the first three entries in the Sequence of Partial Sums $\{ s_n \}$ of this series.

c) Determine whether the sequence of terms converges to 0 (Does $a_k = \frac{k}{(k+1)!} \rightarrow 0$?)

d) Determine whether the series $\sum_{k=1}^{\infty} \frac{k}{(k+1)!}$ converges or diverges .

Solutions to the Example Exercise:

a) The entries in the Sequence of Terms are calculated by the formula: $a_k = \frac{k}{(k+1)!}$

$$a_1 = \frac{1}{(1+1)!} = \frac{1}{2!} = \frac{1}{2\times 1} = \frac{1}{2}$$

$$a_2 = \frac{2}{(2+1)!} = \frac{2}{3!} = \frac{2}{3\times 2\times 1} = \frac{1}{3}$$

$$a_3 = \frac{3}{(3+1)!} = \frac{3}{4!} = \frac{3}{4\times 3\times 2\times 1} = \frac{1}{8}$$

b) The entry in the n^{th}-position of the Sequence of Partial Sums $\{ s_n \}$ is calculated by adding together the first n terms of the [redacted] sequence $\{a_k\}$

$$s_1 = \frac{1}{(1+1)!} = \frac{1}{2} ,$$

$$s_2 = \frac{1}{(1+1)!} + \frac{2}{(2+1)!} = 1 + \frac{1}{3} = \frac{4}{3}$$

$$s_3 = \frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \frac{3}{(3+1)!} = \frac{1}{2} + \frac{1}{3} + \frac{3}{4\times 3\times 2\times 1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{23}{24}$$

c) $$a_k = \frac{k}{(k+1)!} = \frac{k}{(k+1)(k)(k-1)\cdots(1)} = \frac{1}{(k+1)(1)(k-1)\cdots(1)} \rightarrow 0 \quad \text{as } k \rightarrow \infty .$$

Yes, the sequence of terms converges to 0 .

d) Notice that the fact that the sequence of terms converges to 0 , does not tell us whether or not the series is convergent.

Solutions for Exercise #1:

We determine whether the series $\sum_{k=1}^{\infty} \frac{k}{(k+1)!}$ **converges or diverges by using the Ratio Test.**

$$\frac{a_{k+1}}{a_k} = \frac{\frac{k+1}{(k+2)!}}{\frac{k}{(k+1)!}} = \frac{k+1}{(k+2)!} \times \frac{(k+1)!}{k} = \frac{(k+1)!}{(k+2)!} \times \left(\frac{k+1}{k}\right) =$$

$$= \frac{1}{k+2} \times \left(\frac{k\left(1+\frac{1}{k}\right)}{k}\right) = \left(\frac{1}{k+2} \times \left(1+\frac{1}{k}\right)\right) \to L = 0 < 1 \quad as \ k \to \infty .$$

Since the sequence $\left\{\frac{a_{k+1}}{a_k}\right\} \to L < 1$ **(here, L = 0),**

the series $\sum_{k=1}^{\infty} \frac{k}{(k+1)!}$ **converges by the Ratio Test .**

This means that the Sequence of Partial Sums $\{ s_n \}$ **converges to a number and this number is greater than 0 since each** a_k **term is greater than 0..**

Said another way, $\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = \lim_{n \to \infty} s_n > 0$.

However, the Sequence of Terms $a_k = \frac{k}{(k+1)!} \to 0$;

that is, $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k}{(k+1)!} = 0$

Exercise #2: In the following exercises, for the given series $\sum_{k=1}^{\infty} a_k$:

a) Write out the first three entries of the Sequence of Terms, { a_k } .

b) Write out the first three entries of the Sequence of Partial Sums , { s_n } .

c) Determine whether or not the Sequence of Terms { a_k } converges to 0 .

(Does $a_k \to 0$ as $k \to \infty$?)

Exercise:	**Series** $\sum_{k=1}^{\infty} a_k$
1	$\sum_{k=1}^{\infty} \frac{1}{k^3}$
2	$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$
3	$\sum_{k=1}^{\infty} \frac{(k+2)}{(k+1)}$

Solutions for Exercise #2:

1 **For the series** $\sum_{k=1}^{\infty} \frac{1}{k^3}$

1-a) From the Sequence of Terms, { a_k }: $a_1 = 1$, $a_2 = 1/8$, $a_3 = 1/27$.

1-b) From the Sequence of Partial Sums, { s_n }: $s_1 = 1$, $s_2 = 9/8$, $s_3 = 251/216$.

1-c) Yes, as $k \rightarrow \infty$, $a_k = \frac{1}{k^3} \rightarrow 0$.

2 **For the series** $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$

2-a) From the Sequence of Terms, { a_k }: $a_1 = 1/6$, $a_2 = 1/12$, $a_3 = 1/20$.

2-b) From the Sequence of Partial Sums, { s_n }: $s_1 = 1/6$, $s_2 = 3/12$, $s_3 = 18/60$.

2-c) Yes, as $k \rightarrow \infty$, $a_k = \frac{1}{(k+1)(k+2)} \rightarrow 0$.

3 **For the series** $\sum_{k=1}^{\infty} \frac{(k+2)}{(k+1)}$

3-a) From the Sequence of Terms, { a_k }: $a_1 = 3/2$, $a_2 = 4/3$, $a_3 = 5/4$.

3-b) From the Sequence of Partial Sums, { s_n }: $s_1 = 3/2$, $s_2 = 17/6$, $s_3 = 49/12$.

3-c) No, as $k \rightarrow \infty$, $a_k = \frac{(k+2)}{(k+1)} = \frac{k\left(1+\frac{2}{k}\right)}{k\left(1+\frac{1}{k}\right)} = \frac{\left(1+\frac{2}{k}\right)}{\left(1+\frac{1}{k}\right)} \rightarrow 1$.

Thus, the series $\sum_{k=1}^{\infty} \frac{(k+2)}{(k+1)}$ **is divergent by the Divergence Test.**